Stat 565

The PACF & Estimation for ARMA

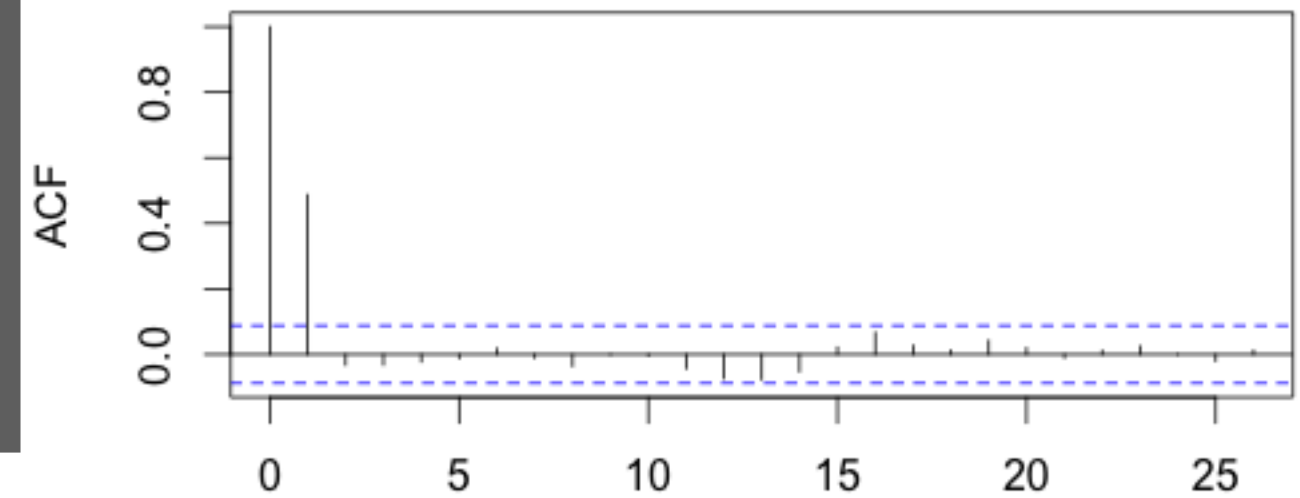
Jan 26 2016

Charlotte Wickham

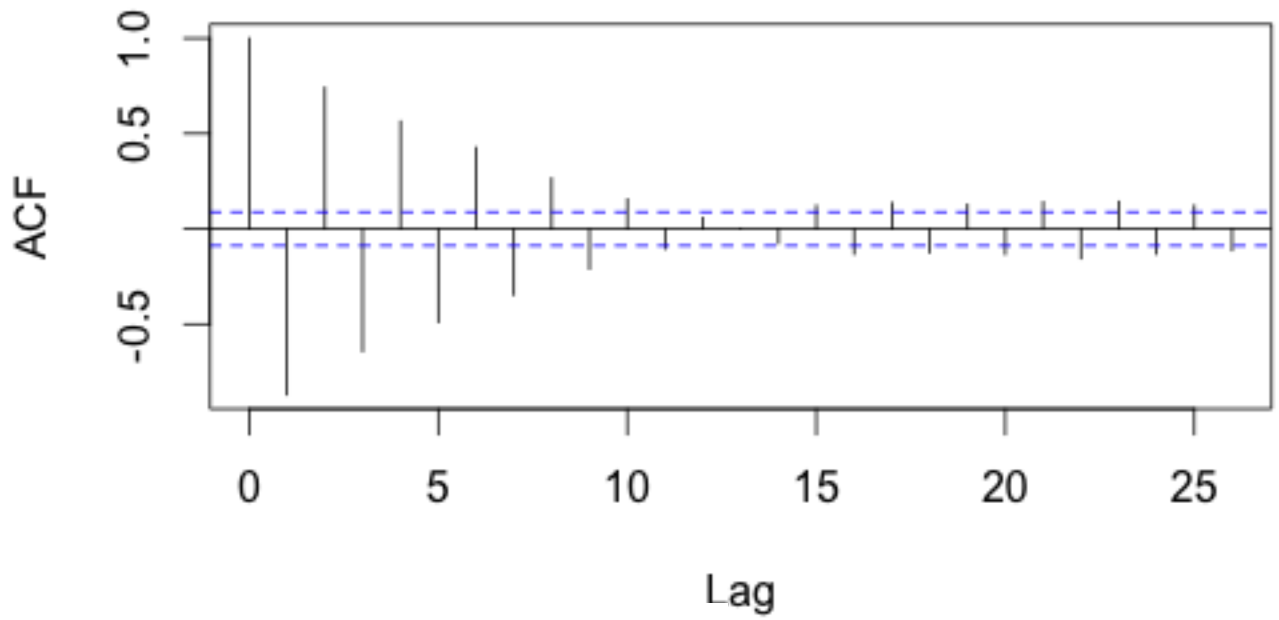
stat565.cwick.co.nz

Which **basic** models might these simulated data come from?
AR(1), MA(1), or white noise

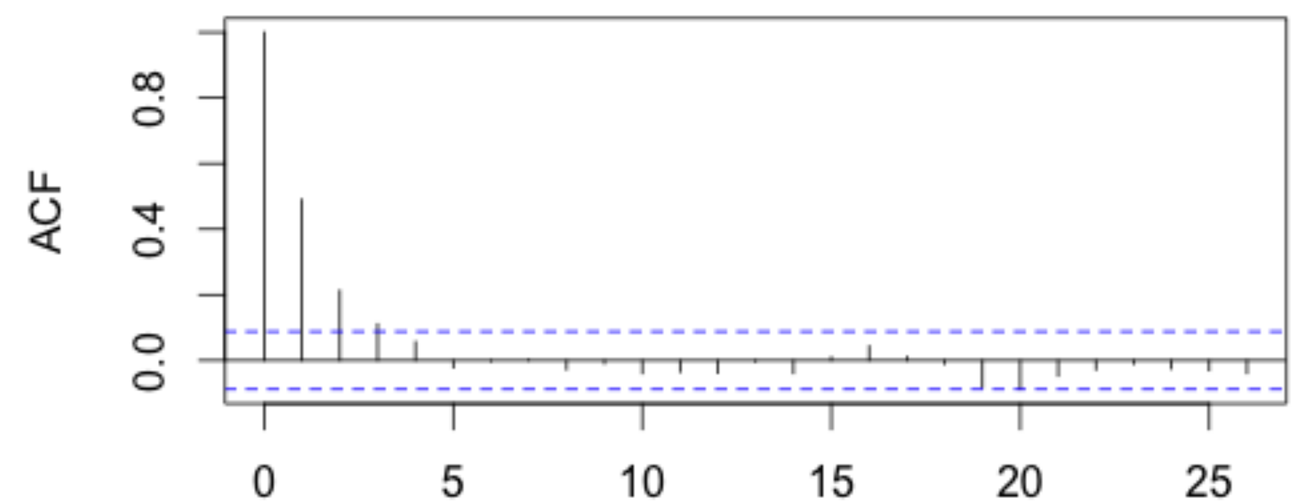
1



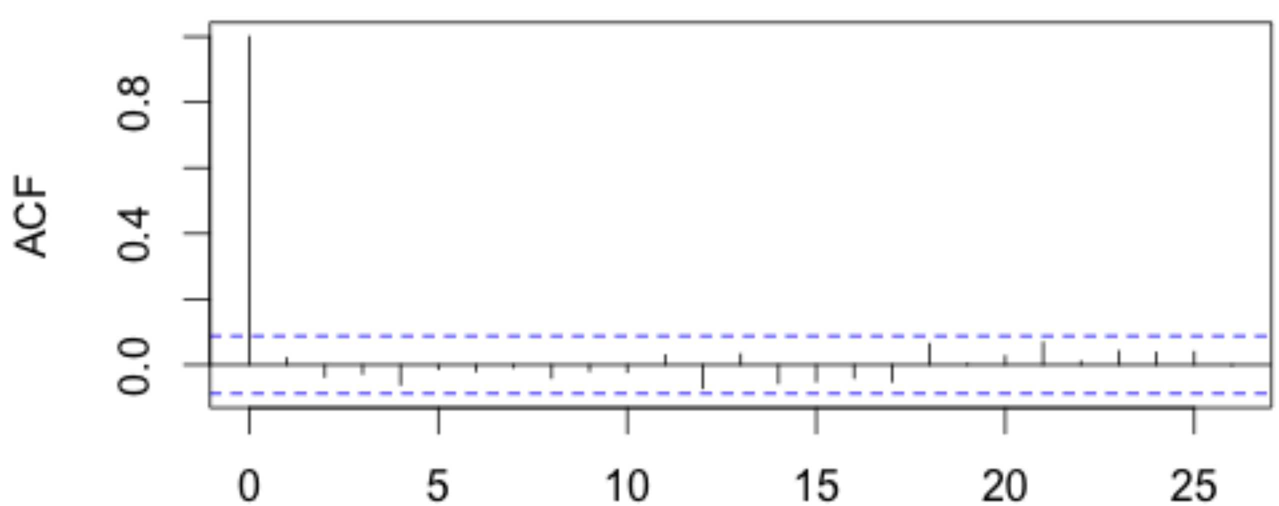
2



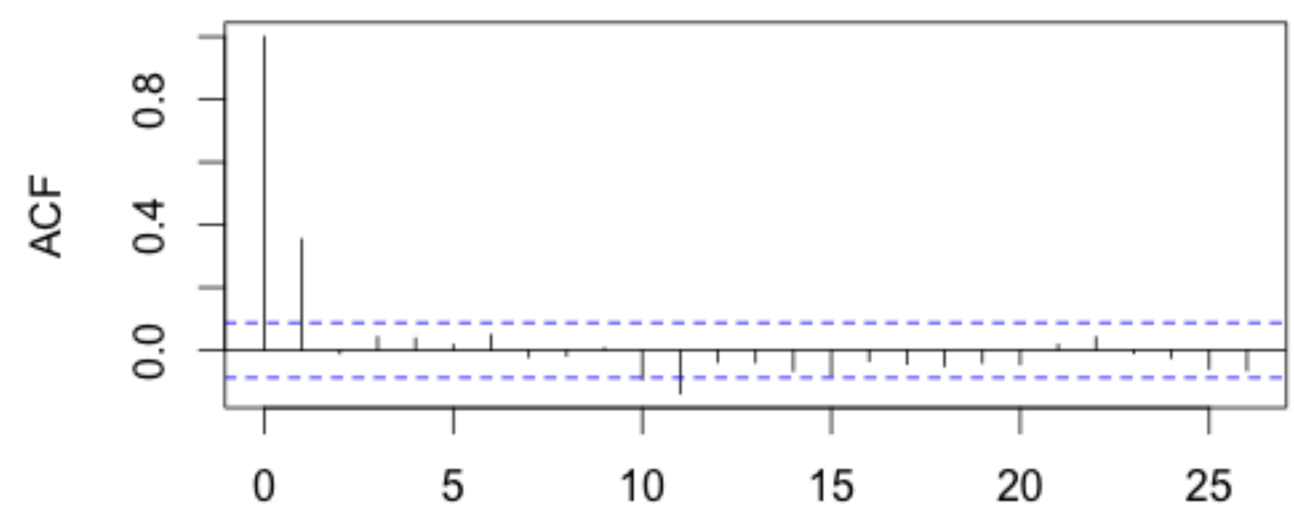
3



4



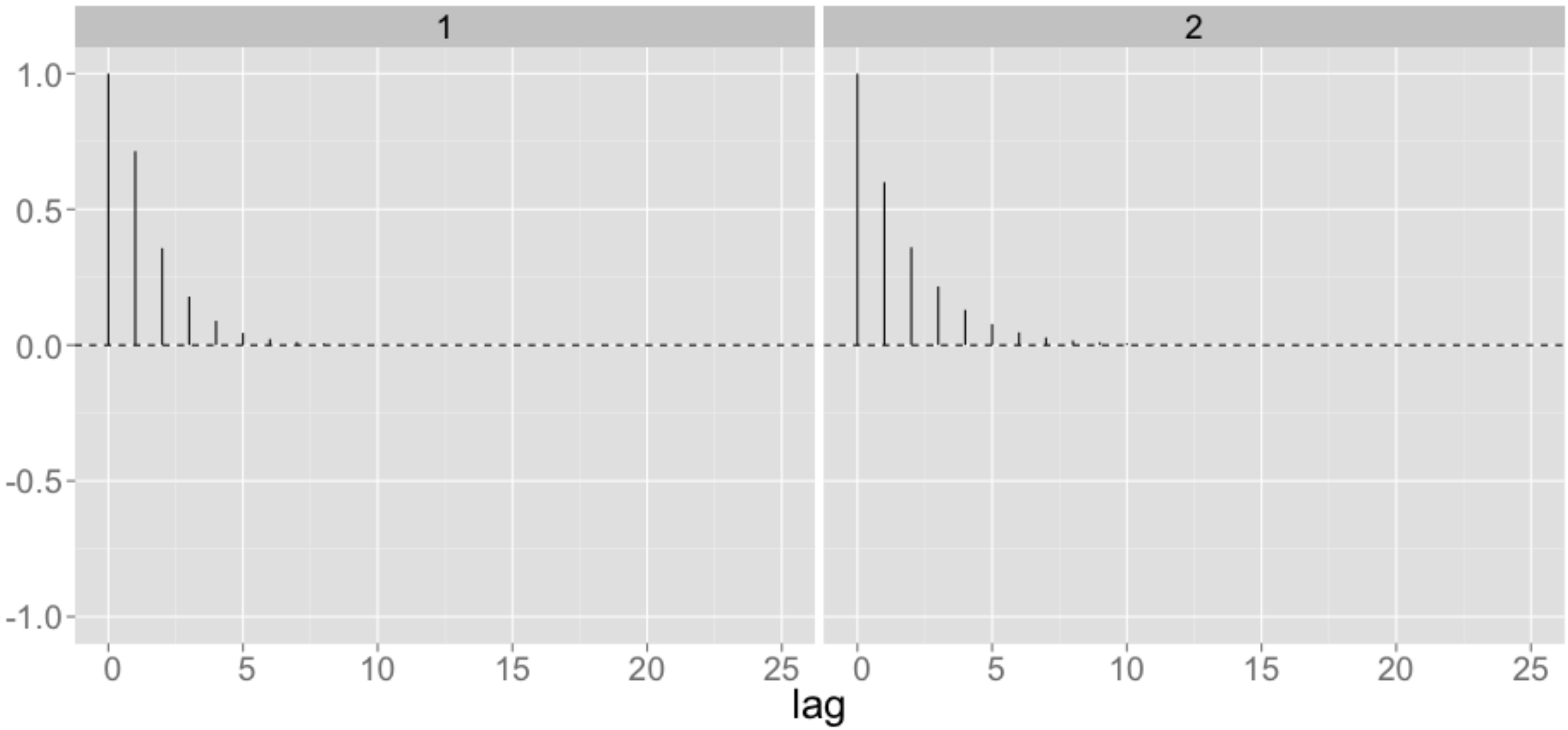
5



ACF

One is AR(1), $\alpha_1 = 0.6$

The other is ARMA(1, 1), $\beta_1 = 0.5$, $\alpha_1 = 0.5$



Which is which?

Partial autocorrelation function

Basic idea: what is the correlation between x_t and x_{t+h}, after taking into account x_{t+1}, x_{t+2}, ..., x_{t+h-1}?

Technically:

Regress x_t on x_{t+1}, x_{t+2}, ..., x_{t+h-1} to find the fitted value $\hat{x}_t$.

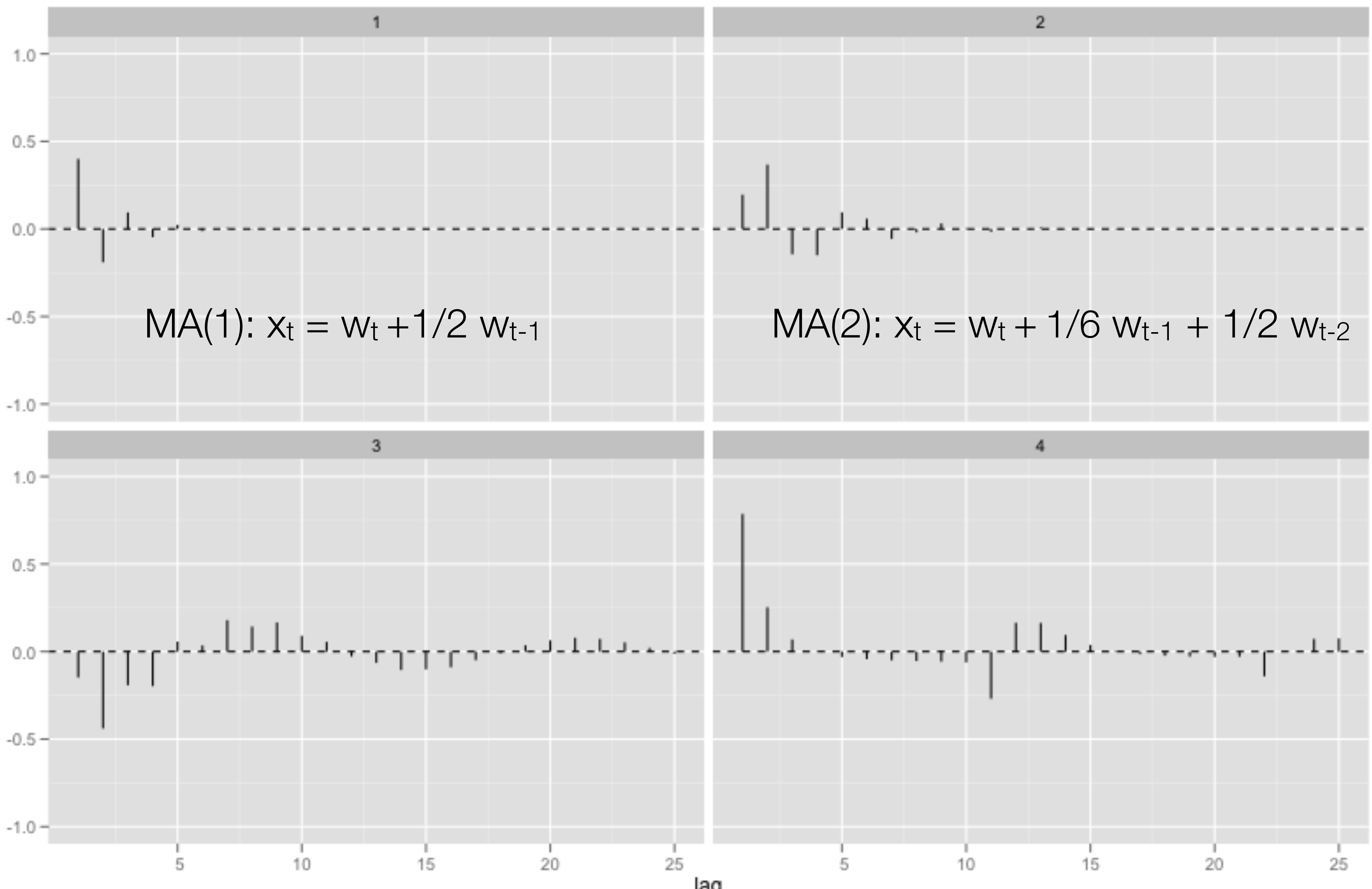
Regress x_{t+h} on x_{t+1}, x_{t+2}, ..., x_{t+h-1} to find the fitted value $\hat{x}_{t+h}$.

Find $\text{cor}(x_t - \hat{x}_t, x_{t+h} - \hat{x}_{t+h})$, call this $\text{PACF}(h) = \phi_{hh}$

Section 3.4 S&S for more detail

PACF

MA(q)



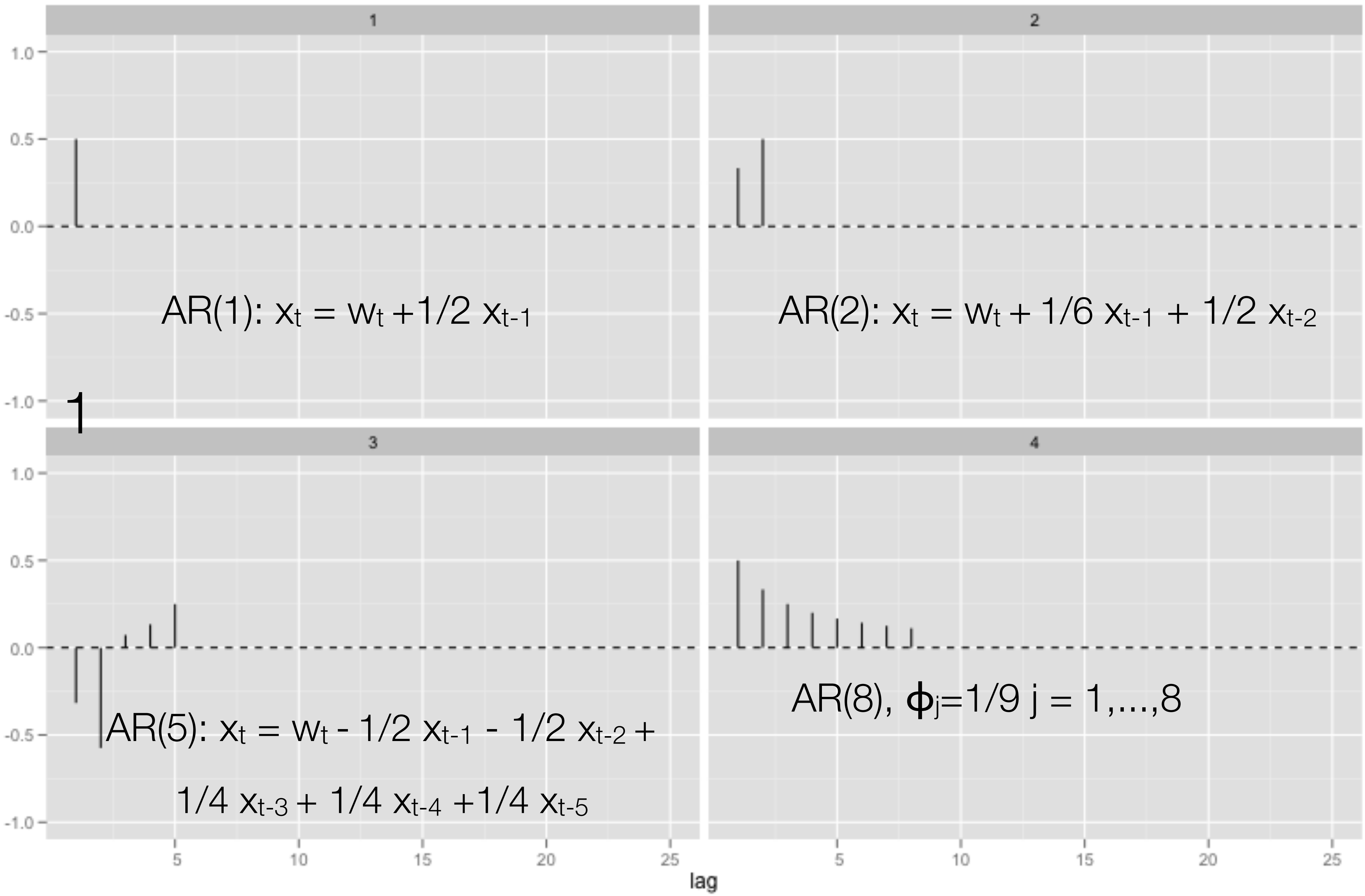
MA(5): $x_t = w_t - 1/2\ w_{t-1} - 1/2\ w_{t-2} +$

$1/4\ w_{t-3} + 1/4\ w_{t-4} + 1/4\ w_{t-5}$

MA(10), $\theta_j = 1/2$ $j = 1,\ldots,10$

PACF

AR(p)

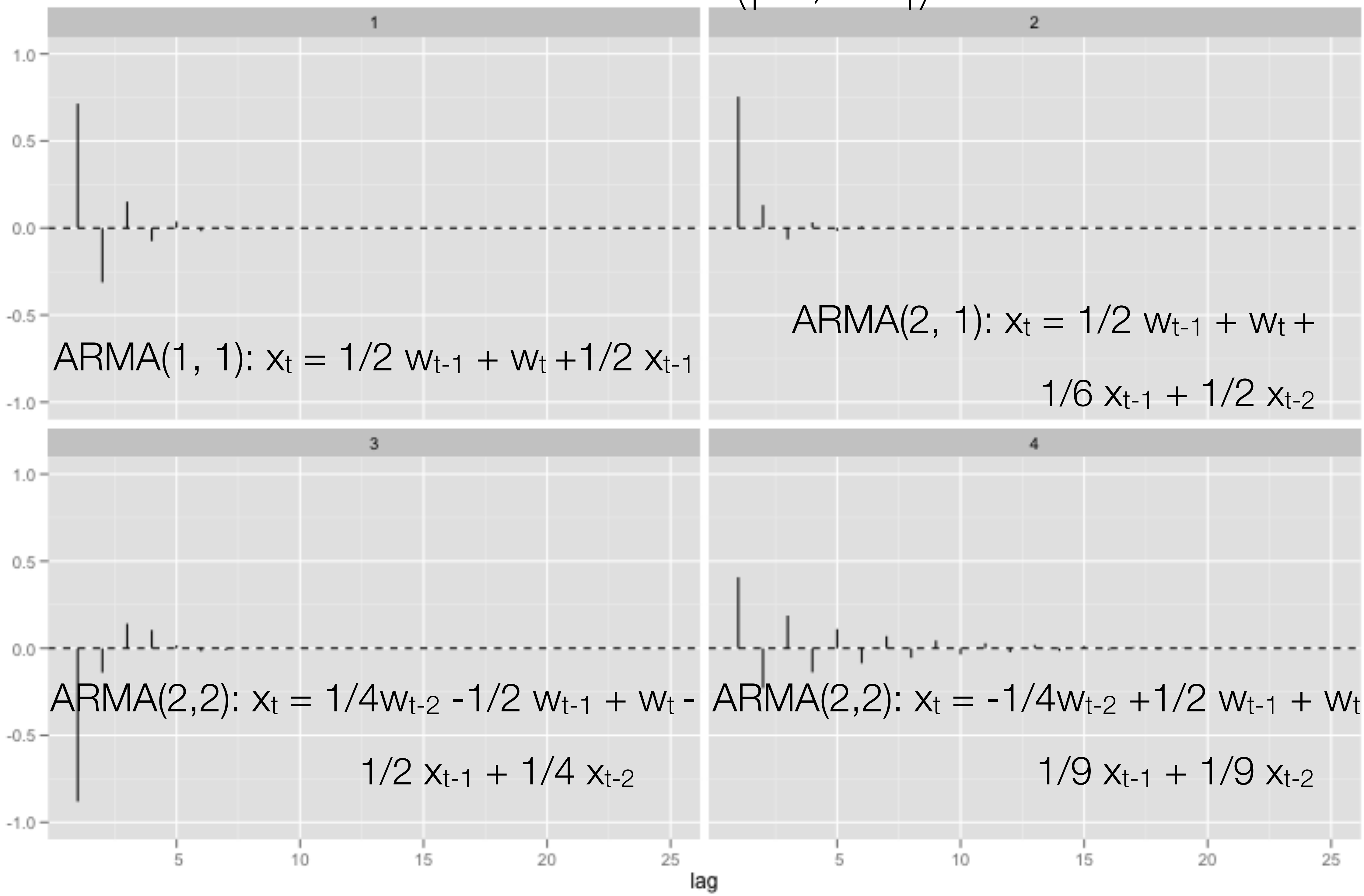


1
2
3
4
1.0
0.5
0.0
-0.5
-1.0
5
10
15
20
25
lag
AR(1): $x_t = w_t + 1/2\ x_{t-1}$
AR(2): $x_t = w_t + 1/6\ x_{t-1} + 1/2\ x_{t-2}$
AR(5): $x_t = w_t - 1/2\ x_{t-1} - 1/2\ x_{t-2} +$
$1/4\ x_{t-3} + 1/4\ x_{t-4} + 1/4\ x_{t-5}$
AR(8), $\phi_j = 1/9\ j = 1,\ldots,8$

1

PACF

ARMA(p, q)



ARMA(1, 1): $x_t = 1/2\ w_{t-1} + w_t + 1/2\ x_{t-1}$

ARMA(2, 1): $x_t = 1/2\ w_{t-1} + w_t + 1/6\ x_{t-1} + 1/2\ x_{t-2}$

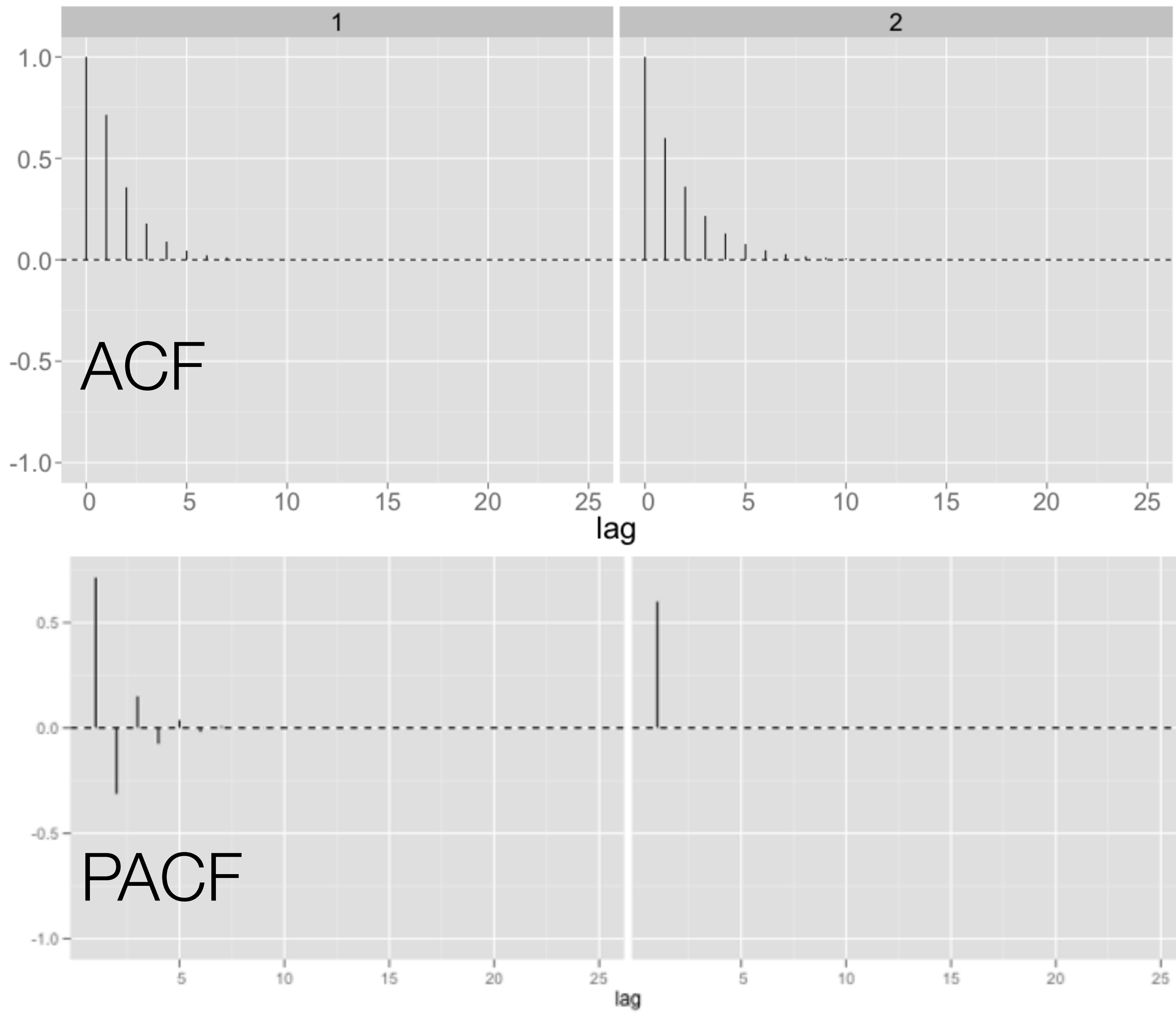
ARMA(2,2): $x_t = 1/4 w_{t-2} - 1/2\ w_{t-1} + w_t - 1/2\ x_{t-1} + 1/4\ x_{t-2}$

ARMA(2,2): $x_t = -1/4 w_{t-2} + 1/2\ w_{t-1} + w_t + 1/9\ x_{t-1} + 1/9\ x_{t-2}$

	MA(q)	AR(p)	ARMA(p, q)
ACF	zero lags > q	tails off	tails off
PACF	tails off	zero lags > p	tails off

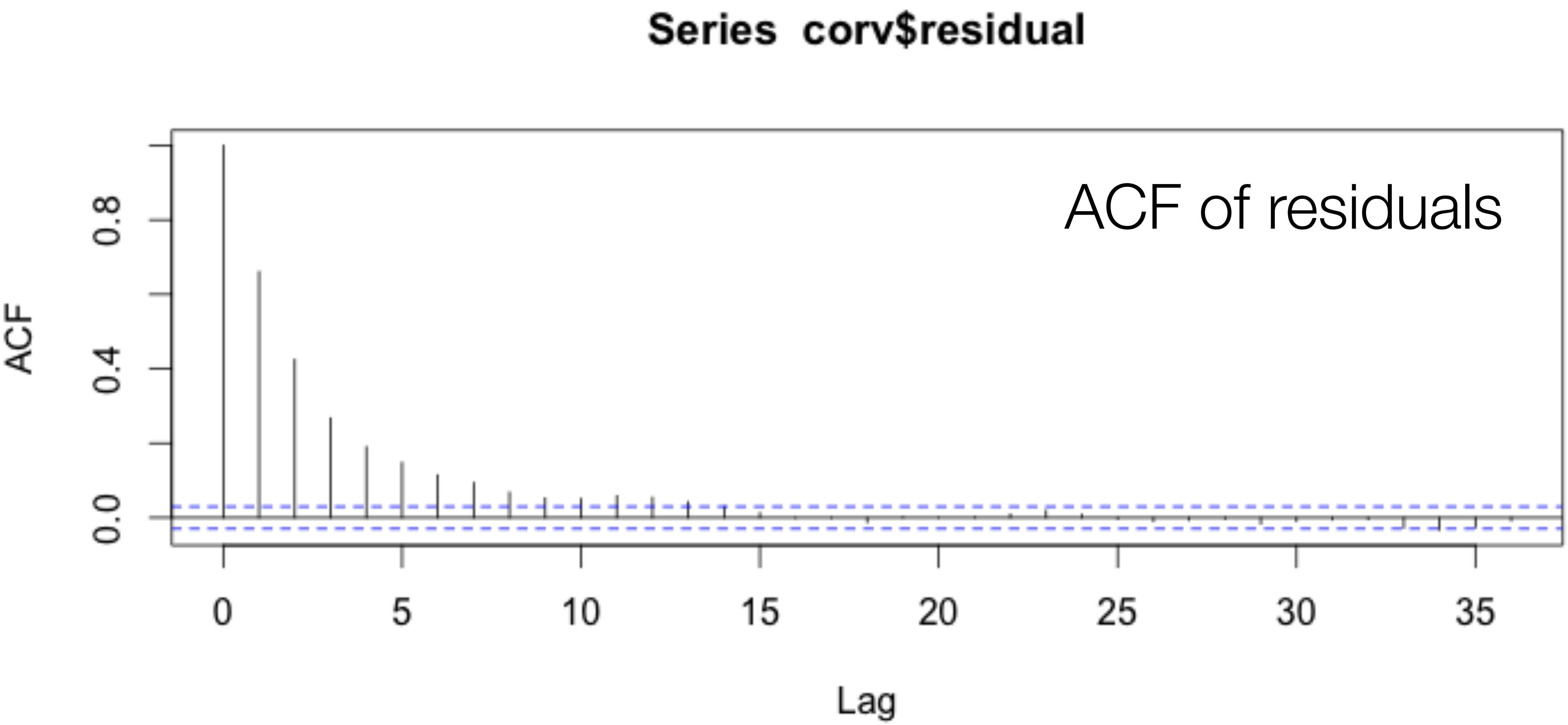
One is AR(1), $\alpha_1 = 0.6$

The other is ARMA(1, 1), $\beta_1 = 0.5$, $\alpha_1 = 0.5$



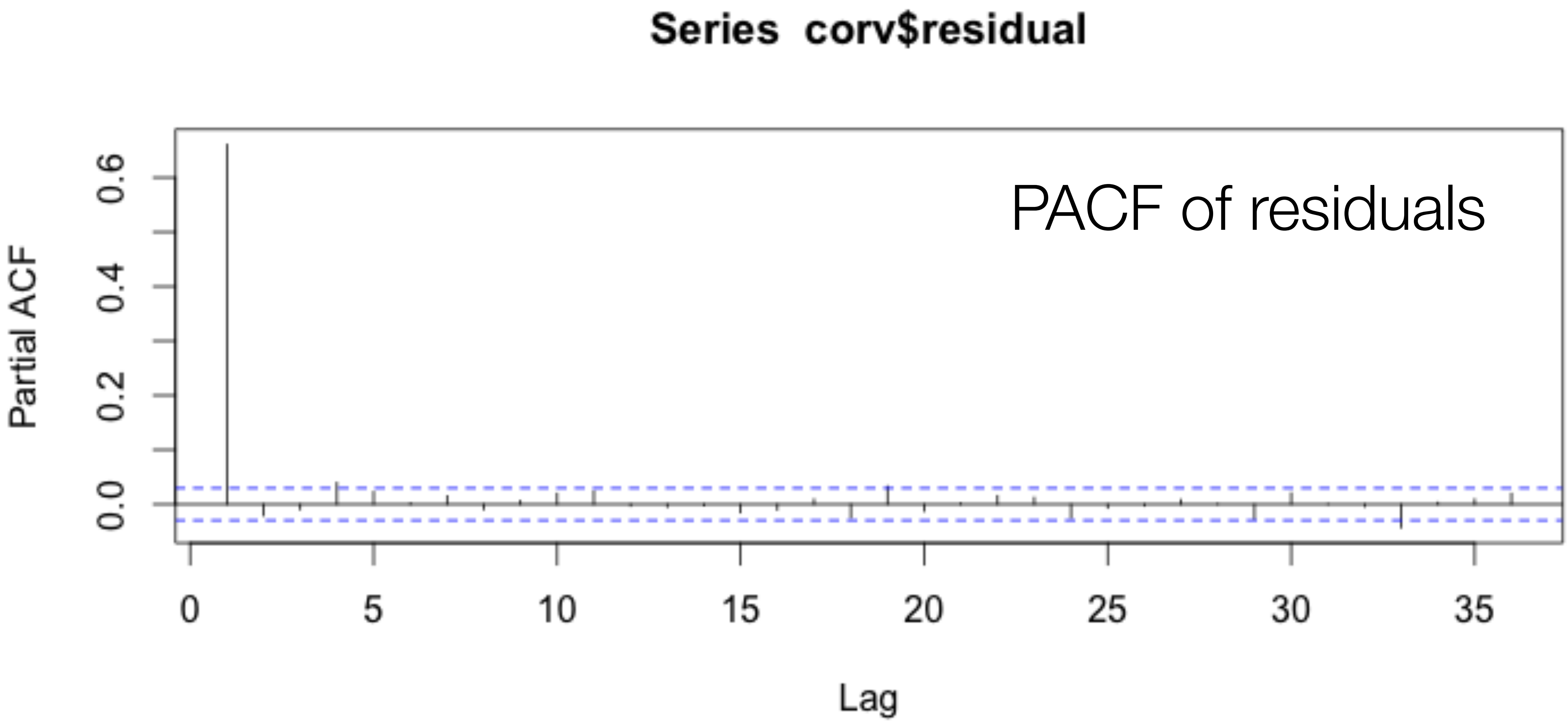
Which is which?

Corvallis temperature



Corvallis temperature

```
pacf(corv$residual, na.action = na.pass)
```



AR(1) looks like a good model

How do we fit it? I.e. how do we estimate α_1?

ARMA(p,q)

Assume we know the order of our process, i.e. we know p and q.

How do we estimate the βs, αs, and σ^2?

Your turn

Name three common approaches to finding an estimate:

Hints:

M   of M

L   S

M   L

The default in R
Has nice properties

M______ __ M___________

We have the theoretical ACF in terms of α, β and σ.

Set the theoretical ACF equal to our sample ACF and solve for the parameters.

So, we don't get too confused, let ρ(h) denote the theoretical ACF, and r(h) the sample ACF.

Corvallis temperature

Assume the residuals can be modelled by an AR(1).

$\rho(1) = \phi$ $\phi = \alpha$, notation slip

$r(1) = \hat{\phi} = 0.6607$

AR(p)

For AR(p) processes we write down a recursion,

$$\rho(h) = \phi_1 \rho(h-1) + \ldots + \phi_p \rho(h-p), \quad h = 1, \ldots, p$$

$$\sigma^2 = \gamma(0) \left(1 - \phi_1 \rho(1) - \ldots - \phi_p \rho(p)\right)$$

The Yule-Walker equations

Derive Yule-Walker eqns

Yule-Walker Estimates

$$\begin{aligned} r(1) &= \hat{\phi}_1 + \hat{\phi}_2 r(1) + \ldots + \hat{\phi}_p r(p) \\ r(2) &= \hat{\phi}_1 r(1) + \hat{\phi}_2 + \ldots + \hat{\phi}_p r(p-2) \\ &\vdots \\ r(p) &= \hat{\phi}_1 r(p-1) + \hat{\phi}_2 r(p-2) + \ldots + \hat{\phi}_p \end{aligned}$$

A set of p equations in p unknowns, solve for $\hat{\phi}_1$ to $\hat{\phi}_p$.

MA(q) and ARMA(p,q)

The method of moments approach gets complicated. End up with non-linear equations to be solved numerically.

The method of moments estimators have bad properties for MA and ARMA processes anyway, so we'll leave it here.

Your turn

Remember linear regression?

$y_i = \beta_0 + \beta_1 x_i + \varepsilon_i$

We can find estimates for β_0 ,β_1 by minimizing

$$\sum_{i=1}^{n} \left(y_i - \hat{\beta}_0 - \hat{\beta}_1 x_i \right)^2$$

L________ S________

Consider the AR(1) process

$x_t = \alpha_1 x_{t-1} + w_t$

Define the residual, $e_t = x_t - \hat{\alpha}_1 x_{t-1}$

We could consider finding the $\hat{\alpha}_1$ that minimises the sum of squared residuals,

$$\sum_{t=2}^{n} \left(x_t - \hat{\alpha}_1 x_{t-1}\right)^2 = \sum_{t=2}^{n} e_t^2$$

called conditional least squares

since we don't see x_0

L_______ s_______ for MA

In general we can always define these residuals, but for MA and ARMA processes they are recursive. For example, MA(1)

$$e_1 = x_1 + \beta_1 e_0$$ assume $e_0 = 0$

$$e_2 = x_2 + \beta_1 e_1$$

$$\vdots$$

$$e_n = x_n + \beta_1 e_{n-1}$$

L________ s__________ in ARMA

For a general ARMA(p,q):

$$e_t = x_t - \alpha_1 x_{t-1} - \ldots - \alpha_p x_{t-p}$$
$$+ \beta_1 e_{t-1} + \ldots + \beta_q e_{t-q}$$

And we have to set $e_p = \ldots = e_{p+1-q} = 0$, and sum starting at t = p, to avoid the x_t we haven't observed

The minimization is done numerically

M_______________ L_________________

Assume a distribution for the white noise (usually Gaussian), then write the joint density function of our data as a function of the parameters, the likelihood,

$$L(\beta, \theta, \sigma^2) = f(x_1, x_2, \dots x_n; \beta, \theta, \sigma^2)$$

Find the parameters that maximise the likelihood.

For the non-statisticans: The joint density, f, tells us the probability of our data given certain parameter values. The likelihood, L, tells us how *likely* certain parameters are given our data (f and L are the same function, we just switch what we consider to be the variable). We estimate the parameters by choosing the most likely parameters given the data we saw.

M_____________ L_______________

Assuming our white noise is Gaussian then the ARMA(p, q), x_t, process is also Gaussian and the likelihood is

$$x_t \sim Normal_n(\mathbf{0}, \Sigma)$$

where $$\Sigma_{ij} = Cov(x_i, x_j) = \gamma(|i - j|)$$

It's complicated, but there are general algorithms for maximizing it.

Maximum Likelihood

The way the function arima in R does it by default.

Nice asymptotic properties, deals with missing data easily.

Always lower variance than method of moments.

Fitting in R

```
arima(ts, order = c(p, d, q))
```

ignore for now

Thursday

I’m out of town.

Chris will lead lecture.

Bring laptops!