General Linear Process

$$x_t = \sum_{i=0}^{\infty} \Psi_i Z_{t-i}$$

MA(1):

$$x_t = \beta_1 Z_{t-1} + Z_t$$

$$= \sum_{i=0}^{\infty} \Psi_i Z_{t-i} \qquad \begin{aligned} \Psi_0 &= 1 \\ \Psi_1 &= \beta_1 \\ \Psi_i &= 0 \quad i \geq 2 \end{aligned}$$

AR(1):

$$x_t = \alpha_1 x_{t-1} + Z_t$$

$$= Z_t + \alpha_1 Z_{t-1} + \alpha_1^2 Z_{t-2} + \ldots$$

$$= \sum_{i=0}^{\infty} \Psi_i Z_{t-i}$$

$$\Psi_i = \alpha_1^i$$

MA(1) Autocovariance function

$$\gamma(h) = \sigma^2 \sum_{i=0}^{\infty} \Psi_{i+h} \Psi_i$$

$$= \sigma^2 \left(\Psi_h \Psi_0 + \Psi_{h+1} \Psi_1 + \Psi_{h+2} \Psi_2 + \ldots \right)$$

$$= \begin{cases} (1 + \beta_1^2)\sigma^2 & h = 0 \\ \beta_1 \sigma^2 & h = 1 \\ 0 & h = 2 \end{cases}$$

Backshift Operator

MA(1):
$$x_t = \beta_1 Z_{t-1} + Z_t$$
$$= \beta_1 B Z_t + Z_t = (\beta_1 B + 1) Z_t$$

AR(1):
$$x_t = \alpha_1 x_{t-1} + Z_t$$
$$= \alpha_1 B x_t + Z_t$$

$$x_t - \alpha_1 B x_t = Z_t$$
$$(1 - \alpha_1 B) x_t = Z_t$$

MA(q):
$$x_t = Z_t + \beta_1 Z_{t-1} + \dots + \beta_k Z_{t-q}$$
$$= Z_t + \beta_1 B Z_t + \beta_2 B^2 Z_t + \dots + \beta_q B^q Z_t$$
$$= (1 + \beta_1 B + \beta_2 B^2 + \dots + \beta_q B^q) Z_t$$

$$\theta(B) = \underbrace{(1 + \beta_1 B + \beta_2 B^2 + \dots + \beta_q B^q)}_{\text{MA operator}}$$

AR(p)
$$\phi(B) x_t = Z_t$$
verify on your own

MA(1) $\beta_1 = 5$

x: $\rho(h) = \begin{cases} 1 & h=0 \\ \frac{5}{26} & h=1 \\ 0 & h \geq 2 \end{cases}$

MA(1) $\beta_1 = \frac{1}{5}$

y: $\rho(h) = \begin{cases} 1 & h=0 \\ \frac{5}{26} & h=1 \\ 0 & h \geq 2 \end{cases}$

Is $x_t = \omega_t + 2\omega_{t-1} + \omega_{t-2}$ investible

$\theta(B)$:

$$x_t = \omega_t + 2B\omega_t + B^2\omega_t$$

write in the form $x_t = \theta(B)\omega_t$

$$= \underbrace{(1 + 2B + B^2)}_{\theta(B)}\omega_t$$

Find roots of $\theta(B)$: for what B, is $\theta(B) = 0$

$$(B+1)(B+1)$$

two roots $B = -1$

$|B| = 1$ not > 1

not investible

2nd example... you do

roots of -6 & -3

$$x_t = \tfrac{1}{2} x_{t-1} - \tfrac{1}{2} w_{t-1} + w_t$$

Looks like ARMA(1,1) since x_t depends on x_{t-1} like AR(1)

w_{t-1} like MA(1)

But,

$$x_t - \tfrac{1}{2} x_{t-1} = -\tfrac{1}{2} w_{t-1} + w_t$$

$$\cancel{\left(1 - \tfrac{1}{2}B\right)} x_t = \cancel{\left(1 - \tfrac{1}{2}B\right)} w_t$$

$$\phi(B)\, x_t = \theta(B)\, w_t$$

$$x_t = w_t$$

i.e. white noise

Find the ACF for $x_t = 0.9x_{t-1} + 0.5w_{t-1} + w_t$

separate x, w terms: $x_t - 0.9x_{t-1} = 0.5w_{t-1} + w_t$

write in terms of B: $\underbrace{(1-0.9B)}_{\phi(B)}x_t = \underbrace{(1+0.5B)}_{\theta(B)}w_t$

We want to write as $x(t) = \sum_{j=0}^{\infty} \psi_j B^j w_{t-j}$

$= \psi(B)w_t$

$$\psi(B) = \frac{\theta(B)}{\phi(B)}$$

rearranging gives $\psi(B)\phi(B) = \theta(B)$

substitute in $(\psi_0 + \psi_1 B + \psi_2 B^2 + \dots)(1-0.9B) = 1 + 0.5B$

& set LHS = RHS constant: $\psi_0 = 1$

B: $\psi_1 B - \psi_0 0.9B = 0.5B$

$\psi_1 - 0.9 = 0.5$

$\Rightarrow \psi_1 = 1.4$

cancel out B and substitute in ψ_0

B^2: $\psi_2 B^2 - 0.9\psi_1 B^2 = 0$

$\Rightarrow \psi_2 = 0.9\psi_1$

B^3: $\psi_3 B^3 - 0.9\psi_2 B^3 = 0$

$\Rightarrow \psi_3 = 0.9\psi_2 = 0.9^2\psi_1$

and so on ... $\psi_k = 0.9^{k-1}\psi_1 = 0.9\,\psi_{k-1}$

$= 0.9^{k-1}(1.4)$ $\quad k>1 \quad \psi_k = 0 \quad k<0$

Now use the general result to find the autocovariance function

$$\gamma(h) = \sigma^2 \sum_{j=0}^{\infty} \psi_{j+h}\psi_j$$

$$\gamma(0) = \sigma^2 \sum_{j=0}^{\infty} \psi_j^2$$

$$= \sigma^2\Big(\underset{\psi_0^2}{1} + \sum_{j=1}^{\infty}\big((0.9)^{k-1}1.4\big)^2\Big)$$

$$= \sigma^2\Big(1 + 1.4^2 \sum_{j=1}^{\infty}(0.9^2)^{k-1}\Big)$$

$$= \sigma^2\Big(1 + \frac{1.4^2}{1-0.9^2}\Big)$$

$$= \sigma^2\, 11.32$$

$$\gamma(1) = \sigma^2 \sum_{j=0}^{\infty}\psi_{j+1}\psi_j \qquad \text{we know for } j>0 \;\; \psi_{j+1} = 0.9\psi_j$$

$$= \sigma^2\Big(\psi_0\psi_1 + \sum_{j=1}^{\infty}\psi_j\, 0.9\,\psi_j\Big)$$

$$= \sigma^2\Big(1(1.4) + 0.9\sum_{j=1}^{\infty}\psi_j^2\Big) \qquad \sum_{j=1}^{\infty}\psi_j^2 = \sum_{j=0}^{\infty}\psi_j^2 - \psi_0^2 = \frac{\gamma(0)}{\sigma^2} - \psi_0^2$$

$$= \sigma^2(1.4) + 0.9\big(\gamma(0) - \psi_0^2\sigma^2\big)$$

$$= \sigma^2(1.4) + 0.9\big(\gamma(0) - \sigma^2\big)$$

$$= \cancel{\sigma^2(0.9\gamma(0) + 0.5)}$$

$$= 0.9\gamma(0) + 0.5\sigma^2$$

$$\gamma(2) = \sigma^2 \sum_{j=0}^{\infty} \psi_{j+2}\psi_j$$ same trick for $j>0$ $\psi_{j+2} = 0.9\,\psi_{j+1}$

$$= \sigma^2 \left(\psi_2\psi_0 + \sum_{j=0}^{\infty} \psi_{j+1}\psi_j\, 0.9 \right)$$

$$= \sigma^2 \left(\psi_2 \right) + 0.9 \left(\gamma(1) - \sigma^2 \psi_0 \psi_1 \right)$$

$$= \sigma^2 \left(0.9(1.4) \right) + 0.9 \left(\gamma(1) - \sigma^2 1.4 \right)$$

$$= \cancel{\ } \; 0.9\, \gamma(1)$$

$$\gamma(3) = \sigma^2 \sum_{j=0}^{\infty} \psi_{j+3}\psi_j$$

$$= \sigma^2 \left(\psi_3\psi_0 + 0.9\left(\gamma(2) - \psi_0\psi_2 \right) \right)$$

$$= \sigma^2 \left(0.9^2\, 1.4 \right) + 0.9 \left(\gamma(2) - \sigma^2 0.9 (1.4) \right)$$

$$= \cancel{\ } \; 0.9\, \gamma(2)$$

$$\gamma(h) = \cancel{\ } \; 0.9\, \gamma(h-1)$$

$$\rho(0) = 1$$

$$\rho(1) = \frac{\gamma(1)}{\gamma(0)} = \frac{0.9\,\gamma(0) + 0.5\sigma^2}{\gamma(0)} = 0.9 + \frac{0.5\sigma^2}{\gamma(0)}$$

$$= 0.9 + \frac{0.5}{11.32}$$

$$\rho(2) = \frac{\gamma(2)}{\gamma(0)} = \frac{0.9\,\gamma(1)}{\gamma(0)} = 0.9\,\rho(1) =$$