Stat 565

Properties Of AR(P) & MA(Q)

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A General Linear Process

A linear process x_t is defined to be a linear combination of white noise variates, Z_t,

$$x_t = \sum_{i=0}^{\infty} \psi_i Z_{t-i}$$

with

$$\sum_{i=0}^{\infty} |\psi_i| < \infty$$

This is enough to ensure stationarity

Autocovariance

One can show that the autocovariance of a linear process is,

$$\gamma(h) = \sigma^2 \sum_{i=0}^{\infty} \psi_{i+h}\psi_i$$

Your turn

Write the MA(1) and AR(1) processes in the form of linear processes.

I.e. what are the ψ_j?

$$x_t = \sum_{i=0}^{\infty} \psi_i Z_{t-i}$$

Verify the autocovariance functions for MA(1) and AR(1)

$$\gamma(h) = \sigma^2 \sum_{i=0}^{\infty} \psi_{i+h} \psi_i$$

Backshift Operator

The **backshift** operator, B, is defined as

$$Bx_t = x_{t-1}$$

It can be extended to powers in the obvious way:

$$B^2x_t = (BB)x_t = B(Bx_t) = Bx_{t-1} = x_{t-2}$$

So, $B^kx_t = x_{t-k}$

Your turn

MA(1): $x_t = \beta_1 Z_{t-1} + Z_t$

AR(1): $x_t = \alpha_1 x_{t-1} + Z_t$

Write the MA(1) and AR(1) models using the backshift operator.

Difference Operator

The **difference** operator, ∇, is defined as,

$$\nabla^d x_t = (1 - B)^d x_t$$

$$(\text{e.g. } \nabla^1 x_t = (1 - B)\, x_t = x_t - x_{t-1})$$

$(1\text{-}B)^d$ can be expanded in the usual way,

$$\text{e.g. } (1 - B)^2 = (1 - B)(1 - B) = 1 - 2B + B^2$$

Some non-stationary series can be made stationary by differencing, see HW#3.

MA(q) process

A moving average model of order q is defined to be,

$$x_t = Z_t + \beta_1 Z_{t-1} + \beta_2 Z_{t-2} + \ldots + \beta_q Z_{t-q}$$

where Z_t is a white noise process with variance σ^2, and the $\beta_1, \ldots, \beta_q$ are parameters.

Can we write this using B?

Moving average operator

$$\theta(B) = 1 + \beta_1 B + \beta_2 B^2 + \ldots + \beta_q B^q$$

Will be important in deriving properties later,....

AR(p) process

An autoregressive process of order p is defined to be,

$$x_t = \alpha_1 x_{t-1} + \alpha_2 x_{t-2} + \ldots + \alpha_p x_{t-p} + Z_t$$

where Z_t is a white noise process with variance σ^2, and the $\alpha_1, \ldots, \alpha_p$ are parameters.

Can we write this using B?

$$\phi(B) = 1 - \alpha_1 B - \alpha_2 B^2 - \ldots - \alpha_p B^p$$

$$\theta(B) = 1 + \beta_1 B + \beta_2 B^2 + \ldots + \beta_q B^q$$
$$\phi(B) = 1 - \alpha_1 B - \alpha_2 B^2 - \ldots - \alpha_p B^p$$

MA(q): $x_t = \theta(B) Z_t$

AR(p): $\phi(B) x_t = Z_t$

Roadmap

Extend AR(1) to AR(p) and MA(1) to MA(q)

Combine them to form ARMA(p, q) processes

Discover a few hiccups, and resolve them.

Then find the ACF (and PACF) functions for ARMA(p, q) processes.

Figure out how to fit a ARMA(p,q) process to real data.

hiccup #1

Your turn

Consider the two MA(1) processes:

$x_t = 5w_{t-1} + w_t$

$y_t = 1/5\ w_{t-1} + w_t$

What are their autocorrelation functions?

$\rho(h) = 1$, when $h = 0$

$= \beta_1/(1 + \beta_1^2)$, $h = 1$

$= 0$, $h \geq 2$

Which one do we choose?

Define an MA process to **invertible** if it can be written,

$$\pi(B)x_t = \sum_{j=0}^{\infty} \pi_j x_{t-j} = w_t$$

an infinite AR process

where $$\pi(B) = \sum_{j=0}^{\infty} \pi_j B^j \text{ and } \sum_{j=0}^{\infty} |\pi_j| < \infty$$

$$x_t = \theta(B) w_t$$
$$\frac{1}{\theta(B)} x_t = w_t$$

Invertible process

For MA(1), the process is invertible if

$| \beta_1 | < 1$.

For MA(q), the process is invertible if the roots of the polynomial $\theta(B)$ all lie outside the unit circle,

i.e. $\theta(z) \neq 0$ for any $| z | \leq 1$.

We will choose to consider only invertible processes

Your turn

Is the MA(2) model,

$x_t = w_t + 2w_{t-1} + w_{t-2}$

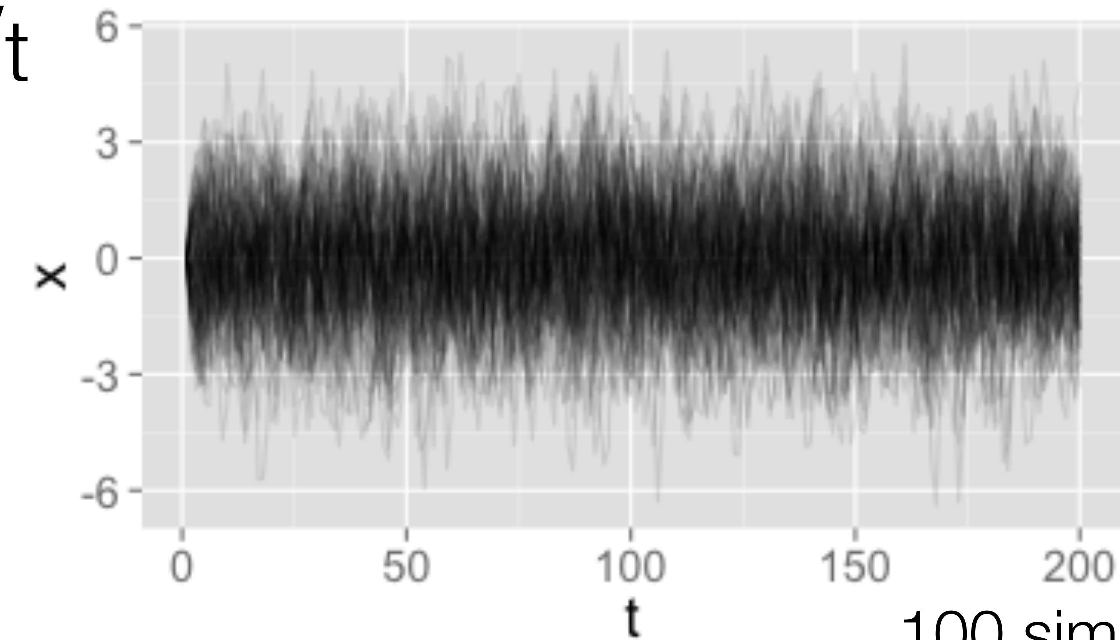
invertible?

What about,

$x_t = w_t + 1/2\ w_{t-1} + 1/18\ w_{t-2}$?

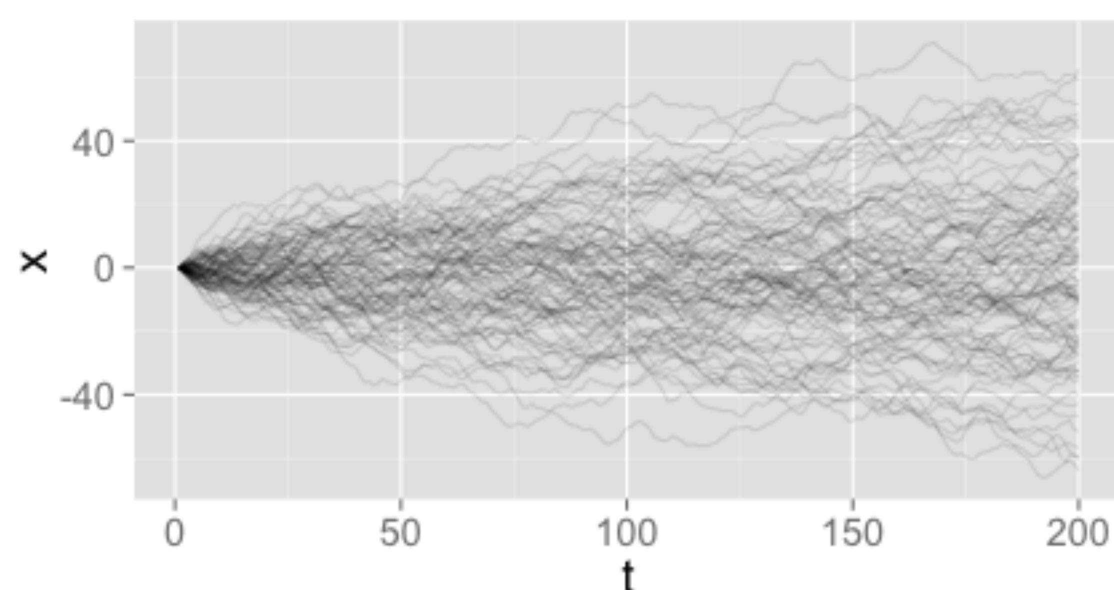
Consider these two AR(2) models

$x_t = x_{t-1} - 1/2x_{t-2} + w_t$



100 sims

$x_t = 1.5x_{t-1} - 1/2x_{t-2} + w_t$



100 sims

$x_0 = x_1 = 0$

hiccup #2: when is an AR(p) stationary?

For AR(1), the process is stationary if $|\alpha_1| < 1$.

For AR(p), the process is stationary if the roots of the polynomial $\phi(B)$ all lie outside the unit circle,

i.e. $\phi(z) \neq 0$ for $|z| \leq 1$.

ARMA(p, q) process

A process, x_t, is ARMA(p,q) if it has the form,

$$\phi(B)\, x_t = \theta(B)\, Z_t,$$

where Z_t is a white noise process with variance σ^2, and

$$\theta(B) = 1 + \beta_1 B + \beta_2 B^2 + \ldots + \beta_q B^q$$
$$\phi(B) = 1 - \alpha_1 B - \alpha_2 B^2 - \ldots - \alpha_p B^p$$

We will assume $Z_t \sim N(0, \sigma^2)$

Properties of ARMA(p,q)

An ARMA(p, q) process is **stationary** if and only if the roots of the polynomial $\phi(z)$ lie outside the unit circle.

I.e. $\phi(z) \neq 0$, for $|z| < 1$

An ARMA(p, q) process is **invertible** if and only if the roots of the polynomial $\theta(z)$ lie outside the unit circle.

I.e. $\theta(z) \neq 0$, for $|z| < 1$

Parameter Redundancy

Example: $x_t = 1/2x_{t-1} - 1/2\, w_{t-1} + w_t$

looks like ARMA(1, 1) but is just white noise.

For an ARMA(p, q) model we assume $\theta(z)$ and $\phi(z)$ have no common factors.

Your turn

Rewrite this ARMA(2, 2) model in a non-redundant form,

$$x_t = -5/6\ x_{t-1} - 1/6\ x_{t-2} + 1/8\ w_{t-2} + 6/8\ w_{t-1} + w_t$$

Finding roots in R

You can check in R:

roots for $\theta(B) = 1 + 1/2B + 1/18B^2$

$$x_t = w_t + 1/2\ w_{t-1} + 1/18\ w_{t-2}$$

```
polyroot(c(1, 1/2, 1/18))
```

check roots have modulus > 1

```
Mod(polyroot(c(1, 1/2, 1/18))) > 1
```

Simulating ARMA(p,q) processes in R

```
?arima.sim
arima.sim(model =
  list(ar = c(0.5, 0.3),
       ma = c(0.5, 0.3),
       sd = 1), 200)
```

α_1, α_2 (ar = c(0.5, 0.3))

β_1, β_2 (ma = c(0.5, 0.3))

σ (sd = 1)   n (200)

Normal white noise process by default

What is the ACF for an ARMA(p, q) process?

It's complicated!

An approach

1. Write the ARMA(p, q) process in the one-sided form.

$$x_t = \sum_{i=0}^{\infty} \psi_i Z_{t-i} = \psi(B) Z_t$$

2. Find the ψ_j by equating coefficients

3. Use the general result for linear processes that,

$$\gamma(h) = \sigma^2 \sum_{i=0}^{\infty} \psi_{i+h} \psi_i$$

Example

What is the ACF of:

$$x_t = 0.9\, x_{t-1} + 0.5\, Z_{t-1} + Z_t$$

More generally. . .

You can set down a recursion for the autocorrelation function and solve it.

That's how `ARMAacf` in R does it:

```
(arma11 <- ARMAacf(ar = 0.9, ma = 0.5,
            lag.max = 25))
qplot(x = 0:25, ymin = 0,
    ymax = arma11, geom = "linerange") +
    geom_hline(yintercept = 0,
      linetype = "dashed") +
    ylim(c(-1, 1))
```

An aside

Sometimes we take a ARMA process,

$$\phi(B)x_t = \theta(B)Z_t$$

where ϕ(B) and θ(B) are finite order polynomials,

And con convert it to an infinite order MA process,

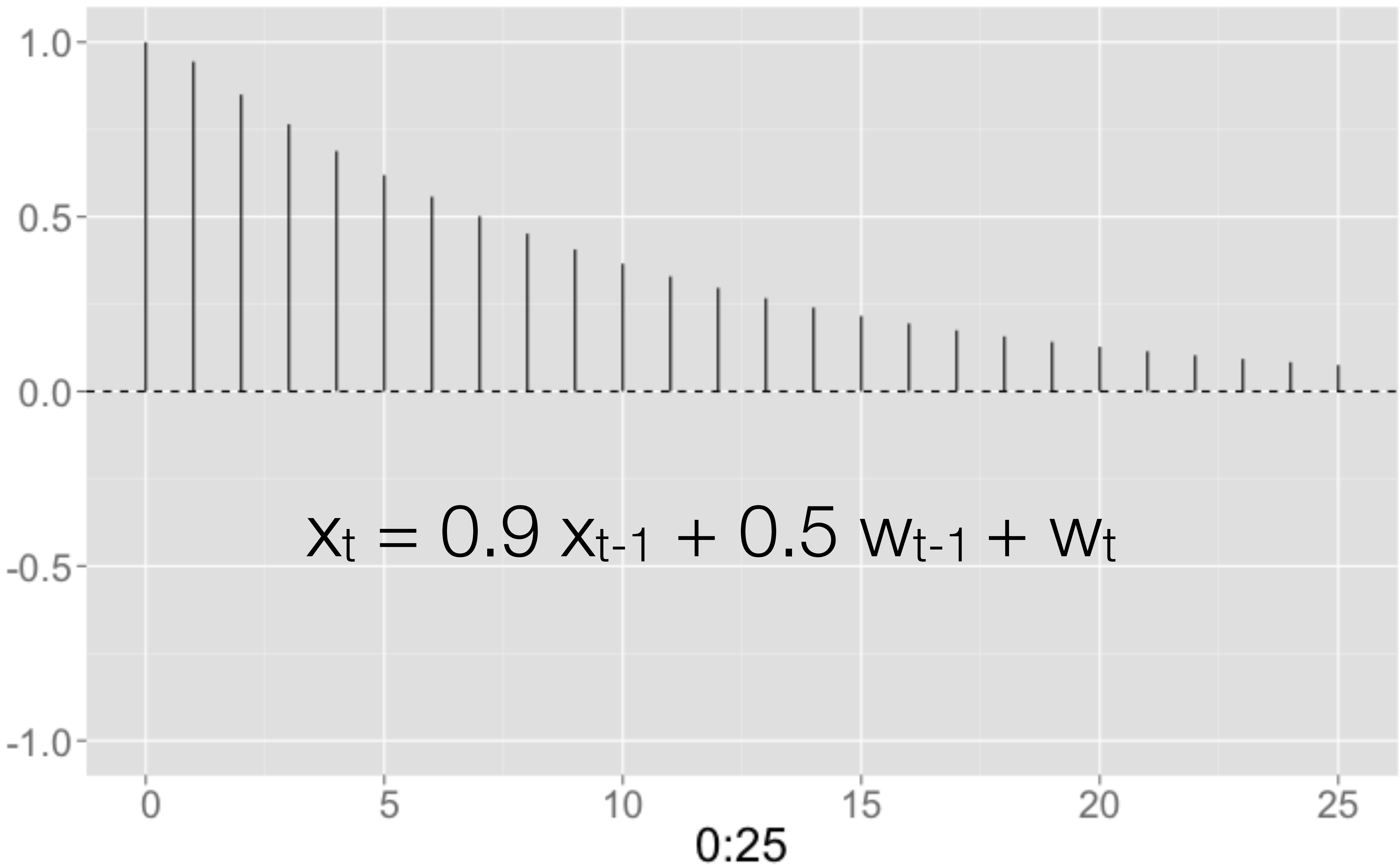
$$x_t = \psi(B)Z_t \qquad \psi(B) = \theta(B)/\phi(B)$$

OR an infinite order AR process,

$$\pi(B)x_t = Z_t \qquad \pi(B) = \phi(B)/\theta(B)$$

ψ, ϕ, π, θ, are pretty consistent notation for these polynomials

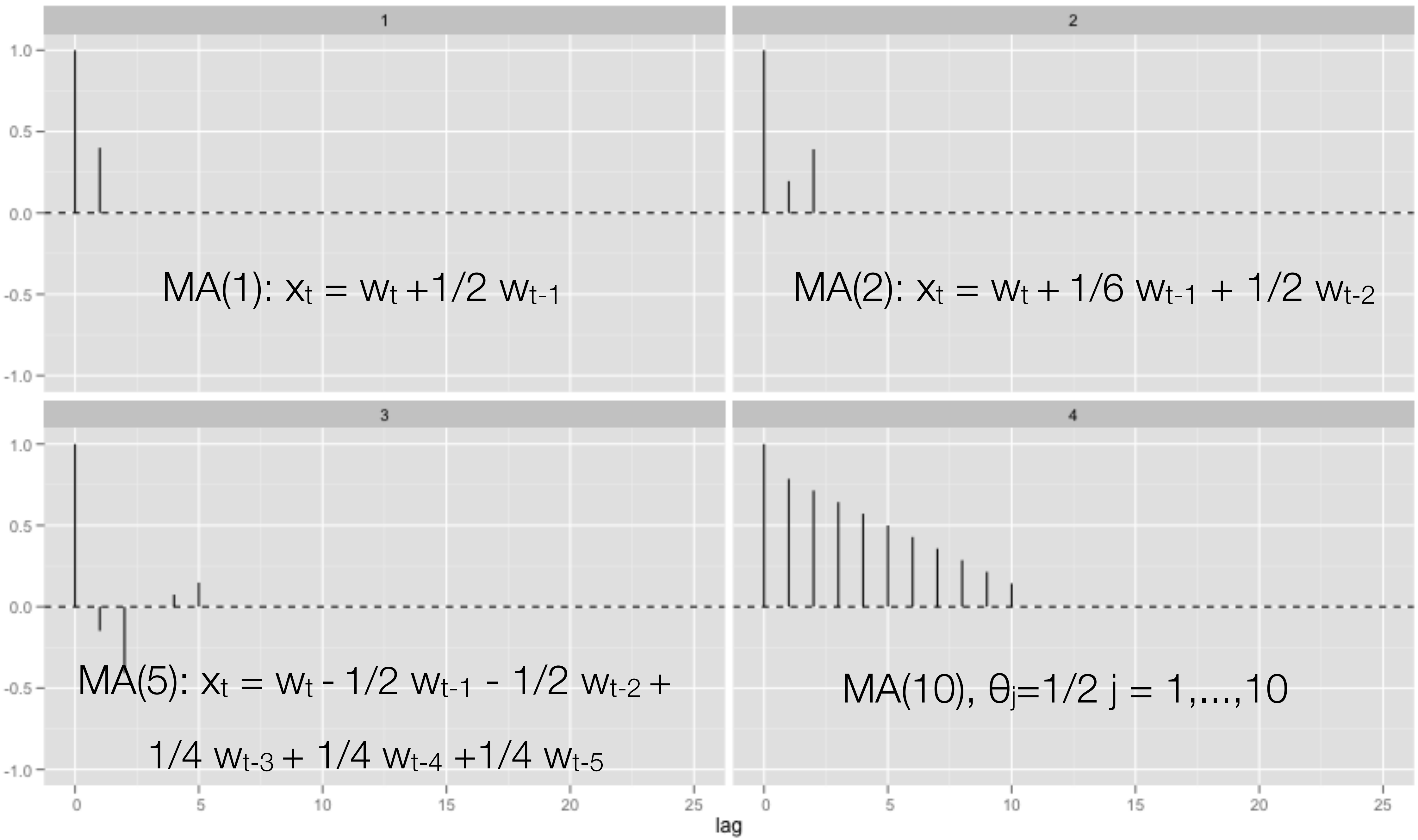
ACF



$$x_t = 0.9\, x_{t-1} + 0.5\, w_{t-1} + w_t$$

ACF

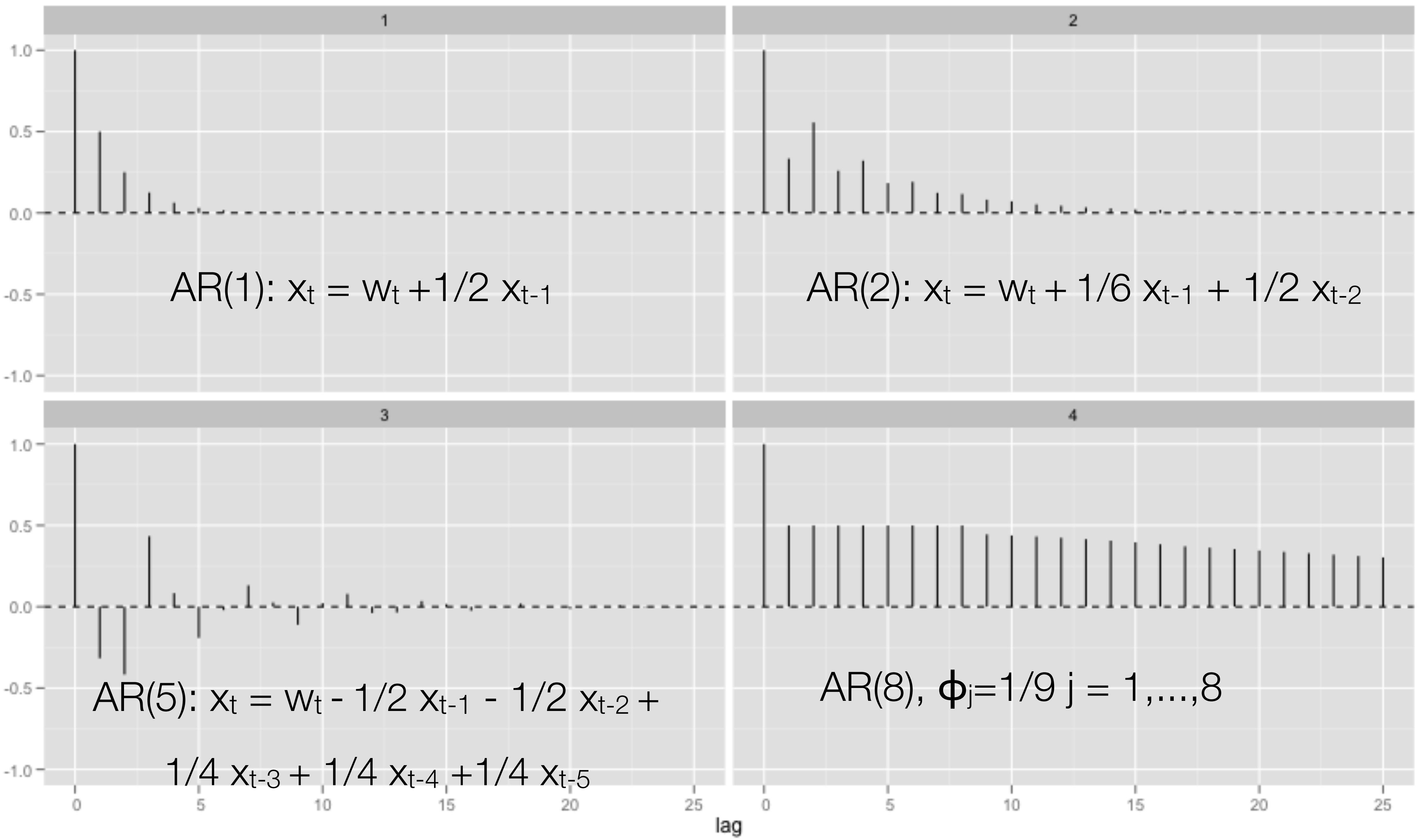
MA(q)



1
2
3
4
1.0
0.5
0.0
-0.5
-1.0
0
5
10
15
20
25
lag
MA(1): $x_t = w_t + 1/2\ w_{t-1}$
MA(2): $x_t = w_t + 1/6\ w_{t-1} + 1/2\ w_{t-2}$
MA(5): $x_t = w_t - 1/2\ w_{t-1} - 1/2\ w_{t-2} +$
$1/4\ w_{t-3} + 1/4\ w_{t-4} + 1/4\ w_{t-5}$
MA(10), $\theta_j = 1/2$ $j = 1,\ldots,10$

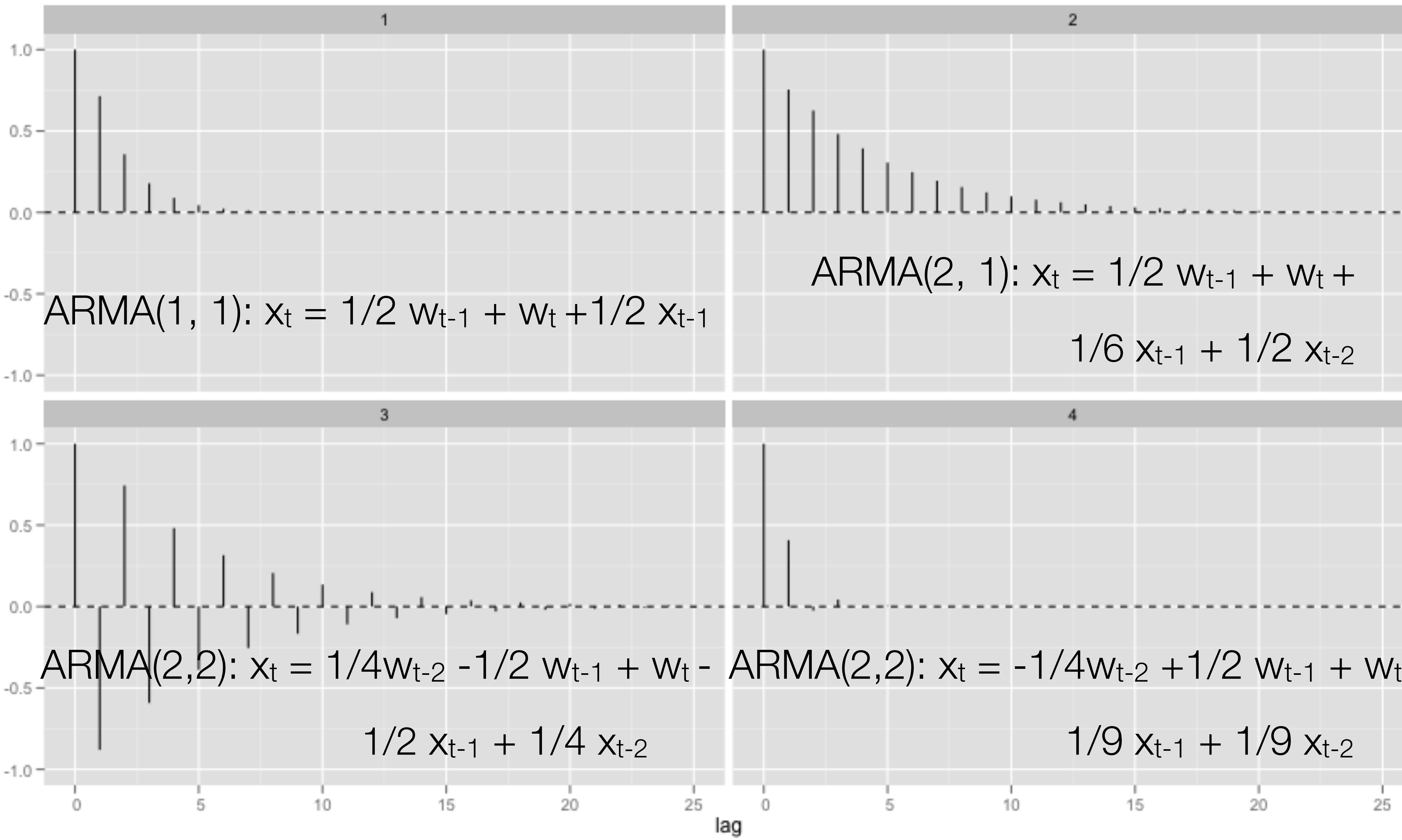
ACF

AR(p)


1
2
3
4
1.0
0.5
0.0
-0.5
-1.0
0
5
10
15
20
25
lag
AR(1): $x_t = w_t + 1/2\ x_{t-1}$
AR(2): $x_t = w_t + 1/6\ x_{t-1} + 1/2\ x_{t-2}$
AR(5): $x_t = w_t - 1/2\ x_{t-1} - 1/2\ x_{t-2} +$
$1/4\ x_{t-3} + 1/4\ x_{t-4} + 1/4\ x_{t-5}$
AR(8), $\phi_j = 1/9\ j = 1,\ldots,8$

ACF

ARMA(p, q)



ARMA(1, 1): $x_t = 1/2\ w_{t-1} + w_t + 1/2\ x_{t-1}$

ARMA(2, 1): $x_t = 1/2\ w_{t-1} + w_t +$

$1/6\ x_{t-1} + 1/2\ x_{t-2}$

ARMA(2,2): $x_t = 1/4 w_{t-2} - 1/2\ w_{t-1} + w_t -$

$1/2\ x_{t-1} + 1/4\ x_{t-2}$

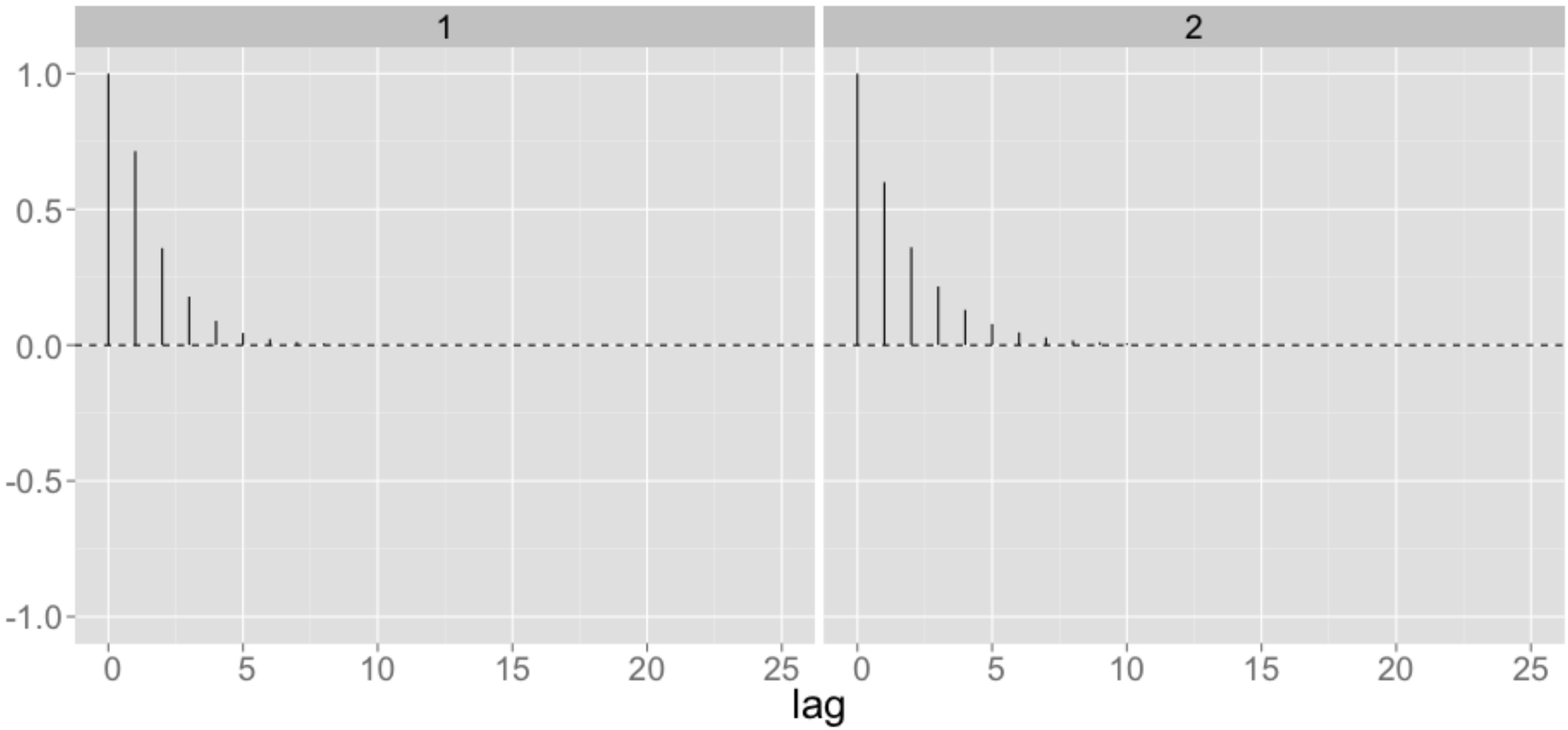
ARMA(2,2): $x_t = -1/4 w_{t-2} + 1/2\ w_{t-1} + w_t +$

$1/9\ x_{t-1} + 1/9\ x_{t-2}$

ACF

One is AR(1), alpha_1 = 0.6

The other is ARMA(1, 1), beta_1 = 0.5, alpha_1 = 0.5



Which is which?