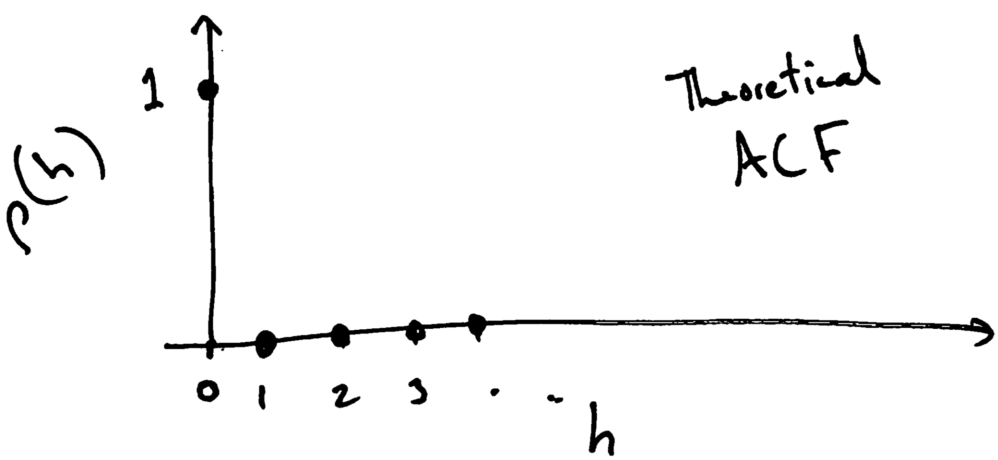
Autocovariance White Noise

$$\gamma(h) = Cov(w_t, w_{t+h}) \longrightarrow E(w_t w_{t-h}) - \underbrace{E(w_t)E[w_{t+h}]}_{\text{often disappear}}$$

$$E(w_t) = 0$$

$$= \begin{cases} Var(w_t) = \sigma^2 & h = 0 \\ 0 & h \neq 0 \end{cases}$$

$$\rho(h) = \frac{\gamma(h)}{\gamma(0)} = \begin{cases} 0 & h \neq 0 \\ 1 & h = 0 \end{cases}$$



Random Walk with drift

$$\mu_t = E(x_t) = E\left(t\delta + \sum_{j=1}^{t} w_j\right)$$

$$= t\delta$$

linearity of expectation
$E[w_j] = 0 \ \forall j$

$$\gamma(t, t+h) = Cov(x_t, x_{t+h})$$

$$= Cov\left(t\delta + \sum_{j=1}^{t} w_j,\ (t+h)\delta + \sum_{j=1}^{t+h} w_j\right)$$

$$= Cov\left(t\delta + w_1 + w_2 + \dots + w_t,\ (t+h)\delta + w_1 + w_2 + \dots + w_t + \dots + w_{t+h}\right)$$

$$\boxed{\begin{aligned} &Cov\left(\sum_{i=1}^{I} y_i, \sum_{j=1}^{J} z_j\right) \\ &= \sum_{i=1}^{I}\sum_{j=1}^{J} Cov(y_i, z_j)\end{aligned}}$$

$$= t\,Var(w_1)$$

$$= t\sigma^2$$

depends on t.

Not stationary

$\delta = 0$. Mean function $= 0 \ \forall t$

even here not stationary...

MA(1)

$$E(x_t) = E(\beta_1 \omega_{t-1} + \omega_t)$$

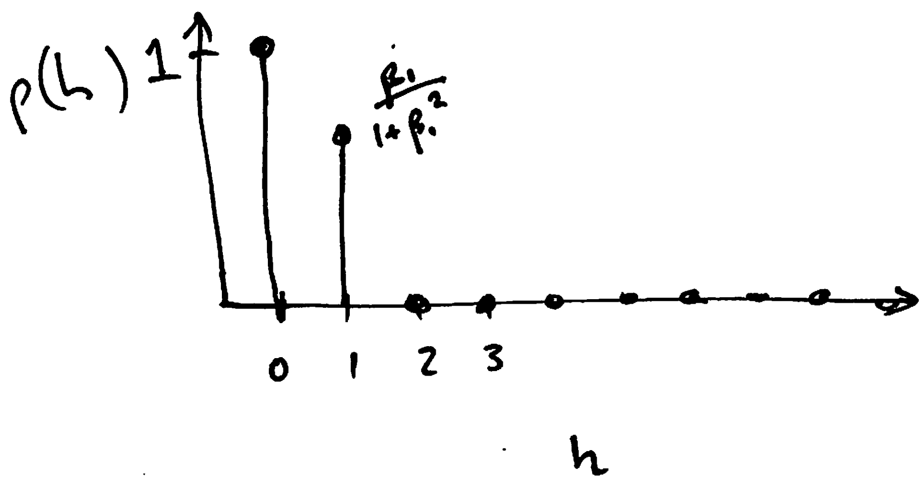
$$= 0$$

$$\gamma(x_t, x_{t+h}) = \mathrm{Cov}(\beta_1 w_{t-1} + \omega_t,\ \beta_1 w_{t+h-1} + w_{t+h})$$

$$= \begin{cases} \beta_1 \sigma^2, & h = 1, -1 \quad |h| = 1 \\ \beta_1^2 \sigma^2 + \sigma^2, & h = 0 \\ 0, & h > \cancel{0}\, 1,\ h < -1 \quad |h| \geq 2 \end{cases}$$

$$\rho(h) = \frac{\gamma(h)}{\gamma(0)} \begin{cases} 1 & h = 0 \\ \dfrac{\beta_1 \sigma^2}{\beta_1^2 \sigma^2 + \sigma^2} = \dfrac{\beta_1}{1 + \beta_1^2} & |h| = 1 \\ 0 & |h| \geq 2 \end{cases}$$

Stationary!


ρ(h)
1
$\frac{\beta_1}{1+\beta_1^2}$
0
1
2
3
h

AR(1)

$$x_t = \alpha_1 x_{t-1} + \omega_t$$

$$E(x_t) = E(\alpha_1 x_{t-1} + \omega_t)$$

$$= E\left(\alpha_1\left(\alpha_1 x_{t-2} + \omega_{t-1}\right) + \omega_t\right)$$

$$= E\left(\alpha_1^2 x_{t-2} + \alpha_1 \omega_{t-1} + \omega_t\right)$$

$$= E\left(\alpha_1^2\left(\alpha_1 x_{t-3} + \omega_{t-2}\right) + \alpha_1 \omega_{t-1} + \omega_t\right)$$

$$= E\left(\omega_t + \alpha_1 \omega_{t-1} + \alpha_1^2 \omega_{t-2} + \alpha_1^3 \omega_{t-3} + \dots\right)$$

$$= 0$$

intuitively pleasant

TRUE

$$E\left(\sum_{i=1}^{\infty} \alpha^{i-1} \omega_{t-i}\right)$$

this isn't always true

$$= \sum_{i=0}^{\infty} E\left[\alpha^{i-1} \omega_{t-i}\right]$$

$$\mathrm{Cov}(x_t, x_{t+h}) = \mathrm{Cov}\Big(w_t + \alpha_1 w_{t-1} + \alpha_1^2 w_{t-2} + \cdots\cdots,$$

$$w_{t+h} + \alpha_1 w_{t+h-1} + \alpha_1^2 w_{t+h-2} + \cdots$$

$$+ \alpha_1^h w_t + \alpha_1^{h+1} w_{t-1} + \cdots\Big)$$

$$= \alpha_1^h \sigma^2 + \alpha_1^{h+2} \sigma^2 + \alpha_1^{h+4} \sigma^2 + \cdots\cdots$$

$$= \alpha_1^h \sigma^2 \sum_{i=0}^{\infty} \alpha^{2i}$$

"geometric series" doesn't always converge

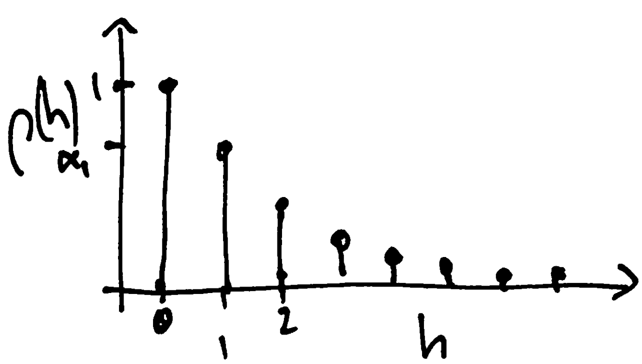
will if $|\alpha^2| < 1$

converges to $\frac{1}{1-\alpha^2}$

$$= \frac{\alpha_1^h \sigma^2}{1-\alpha_1^2} \quad \text{if } |\alpha_1^2| < 1$$

$\iff |\alpha_1| < 1$ weakly stationary

$$\cancel{\gamma(h)} \quad \rho(h) = \frac{\gamma(h)}{\gamma(0)} = \begin{cases} 1 & h = 0 \\ \alpha_1^{|h|} & |h| > 0 \end{cases}$$



$|\alpha_1| < 1$