Stat 565

Some Basic Time Series Models

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Weak Stationarity

A time series $\{x_t\}$ is **weakly stationary** if it's mean function doesn't depend on time, and it's autocovariance function only depends on the distance between the two time points,

$\mu_t = E[x_t] = \mu$

$\Upsilon(s, t) = Cov(x_s, x_t) = \Upsilon(t - s)$

Often rewrite as
$\Upsilon(h) = Cov(x_t, x_{t+h})$

x_t assumed to have finite variance

Autocorrelation

For a stationary process the autocorrelation is:

$$\mathrm{Cor}(x_t, x_{t+h}) = \rho(h) = \Upsilon(h) / \Upsilon(0)$$

Some basic models

White noise

Random walk with drift

Moving average of order 1 MA(1)

Autoregressive of order 1 AR(1)

What is the mean function?
What is the autocovariance function?
Is the process weakly stationary?

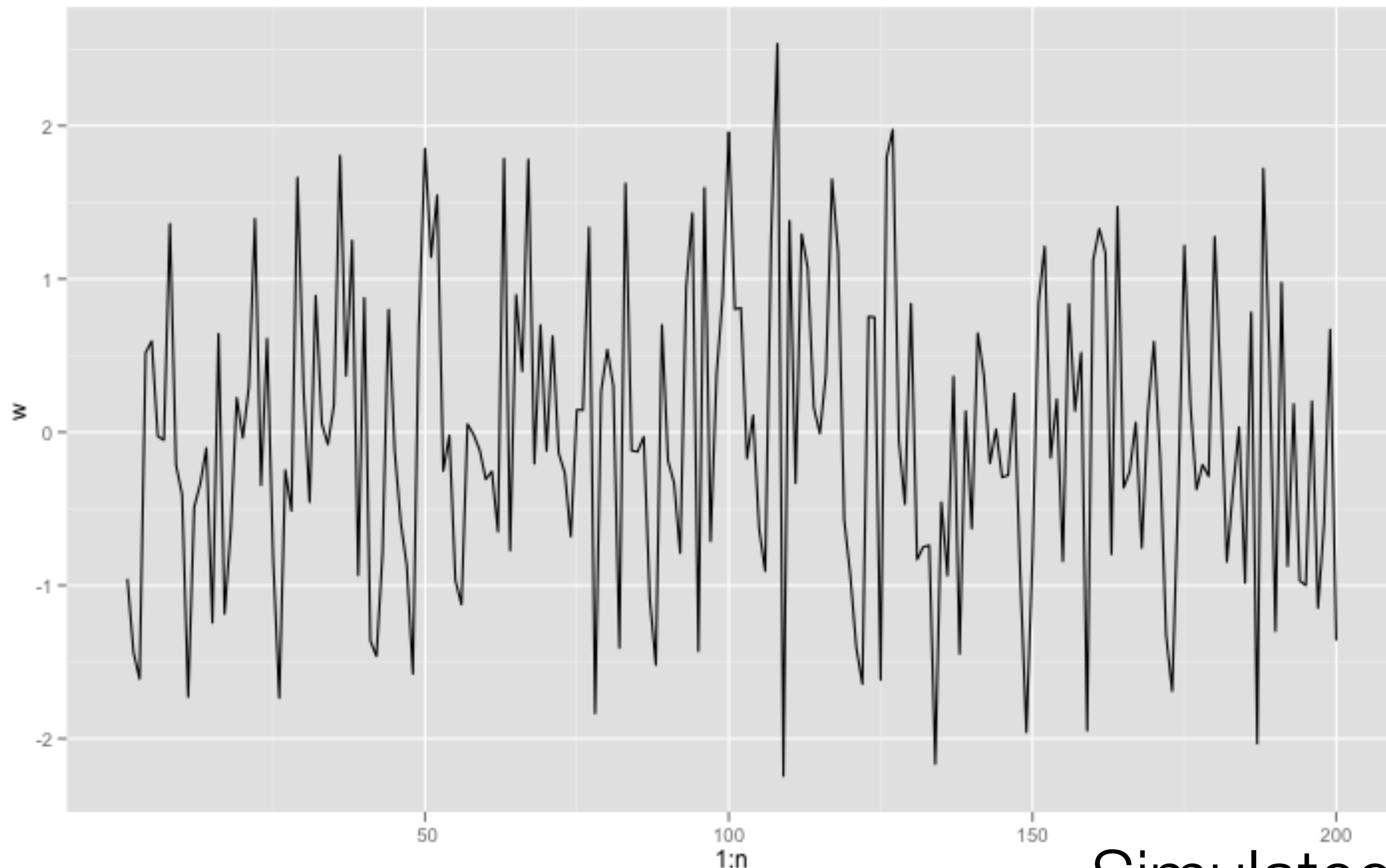
White noise

$\{ w_t \}$ is a white noise process if w_t are **uncorrelated** identically distributed random variables with

$E[w_t] = 0$ and $Var[w_t] = \sigma^2$,   for all t

If the w_t are Normally (Gaussian) distributed, the series is known as Gaussian white noise.

White noise



Simulated

$$\sigma^2 = 1$$

White noise

What is the mean function?

$\mu_t = E[w_t] = 0$

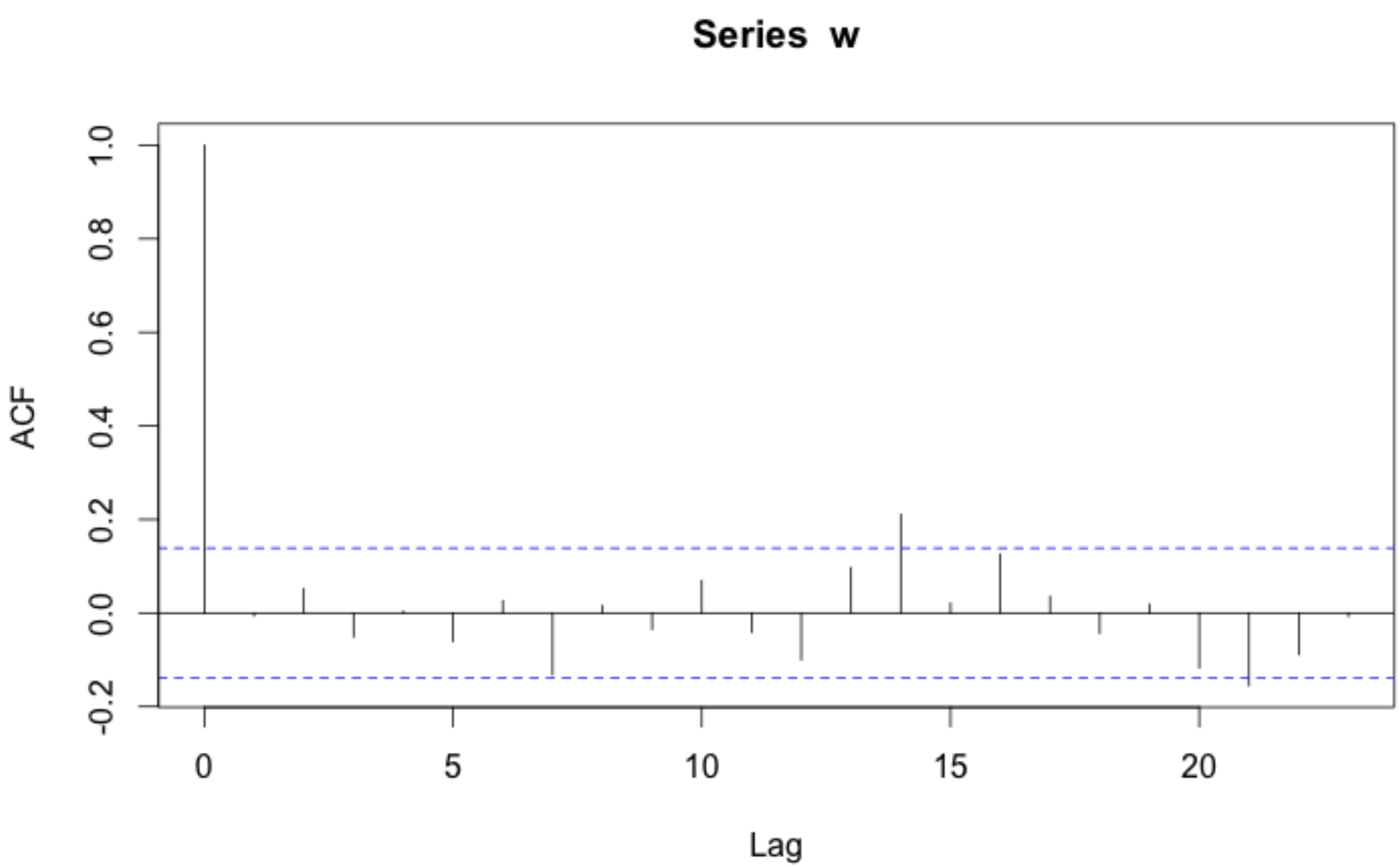
What is the autocovariance function?

$\Upsilon(h) = \{ \sigma^2, h = 1$

$\quad\quad\quad \{ 0, \; \text{otherwise}$

Is white noise stationary?

Yes.


Series w
ACF
1.0
0.8
0.6
0.4
0.2
0.0
-0.2
0
5
10
15
20
Lag

Random walk with drift

drift, a constant

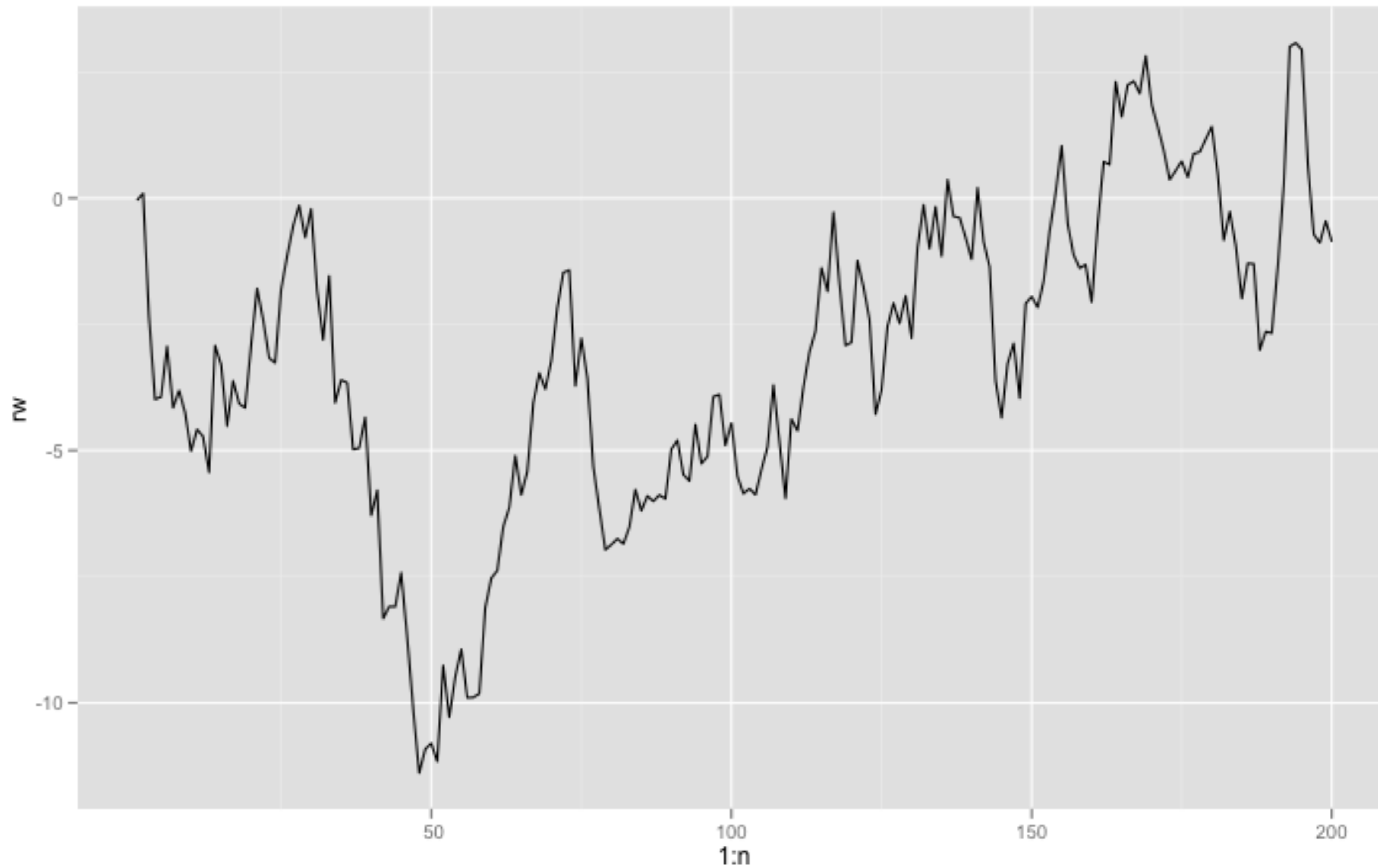
$$x_t = \delta + x_{t-1} + w_t$$

where $\{w_t\}$ is a white noise process, and $x_0 = 0$.

Can rewrite as:

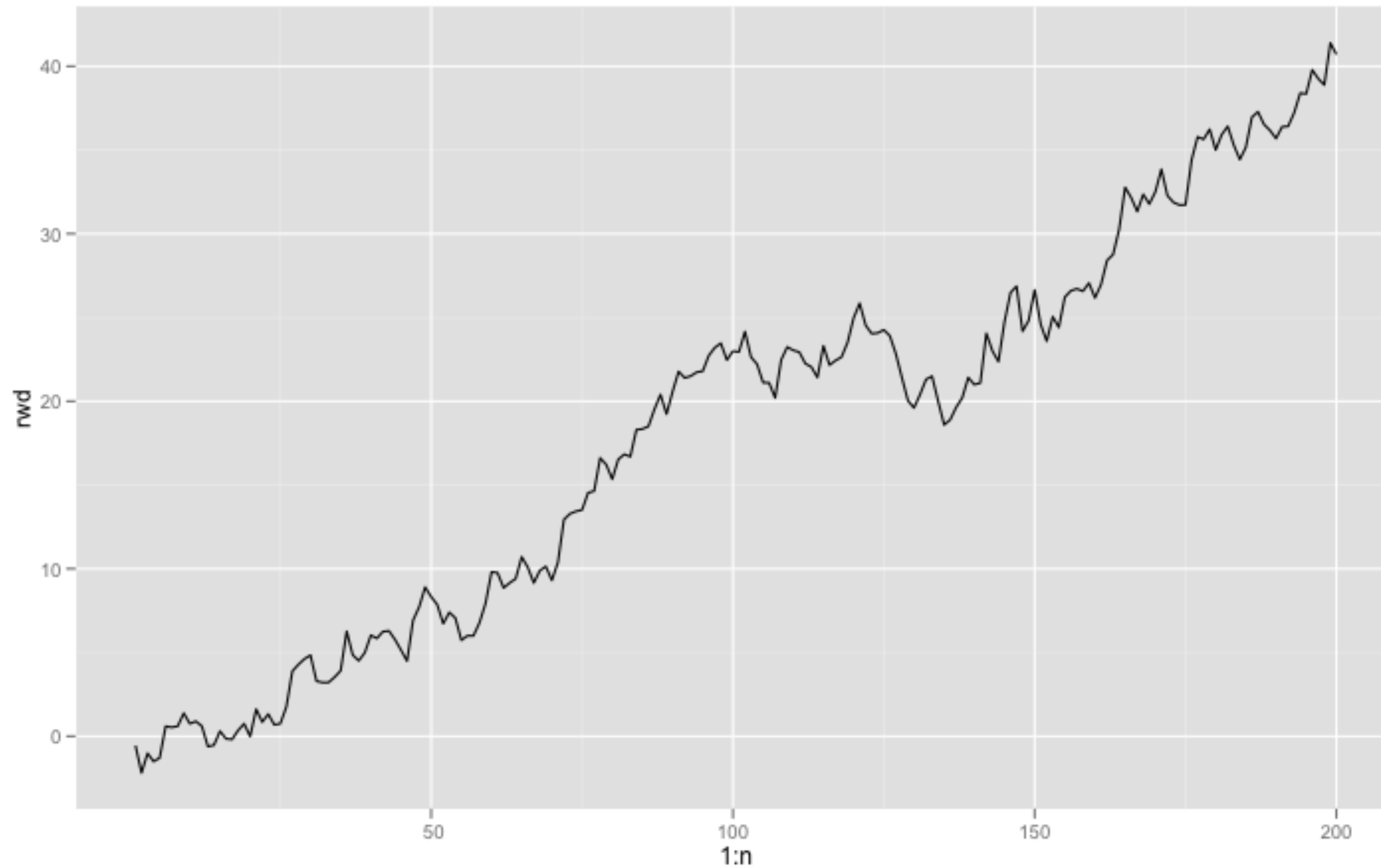
$$x_t = t\delta + \sum_{j=1}^{t} w_j$$

Random walk (drift = 0)



Simulated

Random walk (drift = 0.1)



Simulated

Your turn

Random walk with drift $x_t = t\delta + \sum_{j=1}^{t} w_j$

What is the mean function?

$\mu_t = E[w_t] = ?$

What is the autocovariance function?

$\gamma(t, t+h) = Cov(w_t, w_{t+h})?$

Is the random walk model stationary?

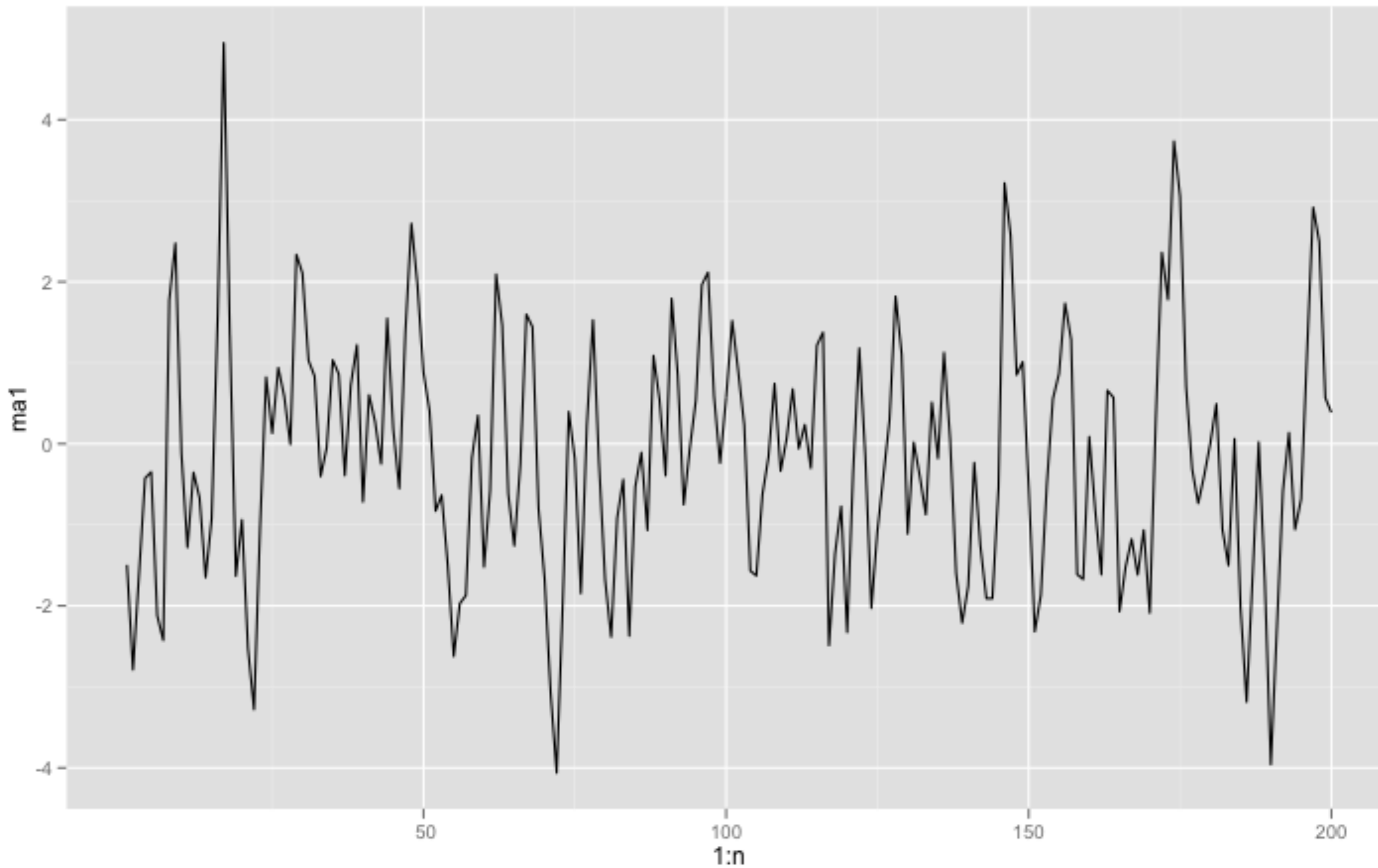
Moving average MA(1)

$$x_t = \beta_1 w_{t-1} + w_t$$

where $\{w_t\}$ is a white noise process.

We'll see higher order MA processes later...

MA(1) $\beta_1 = 1$



Simulated

Your turn

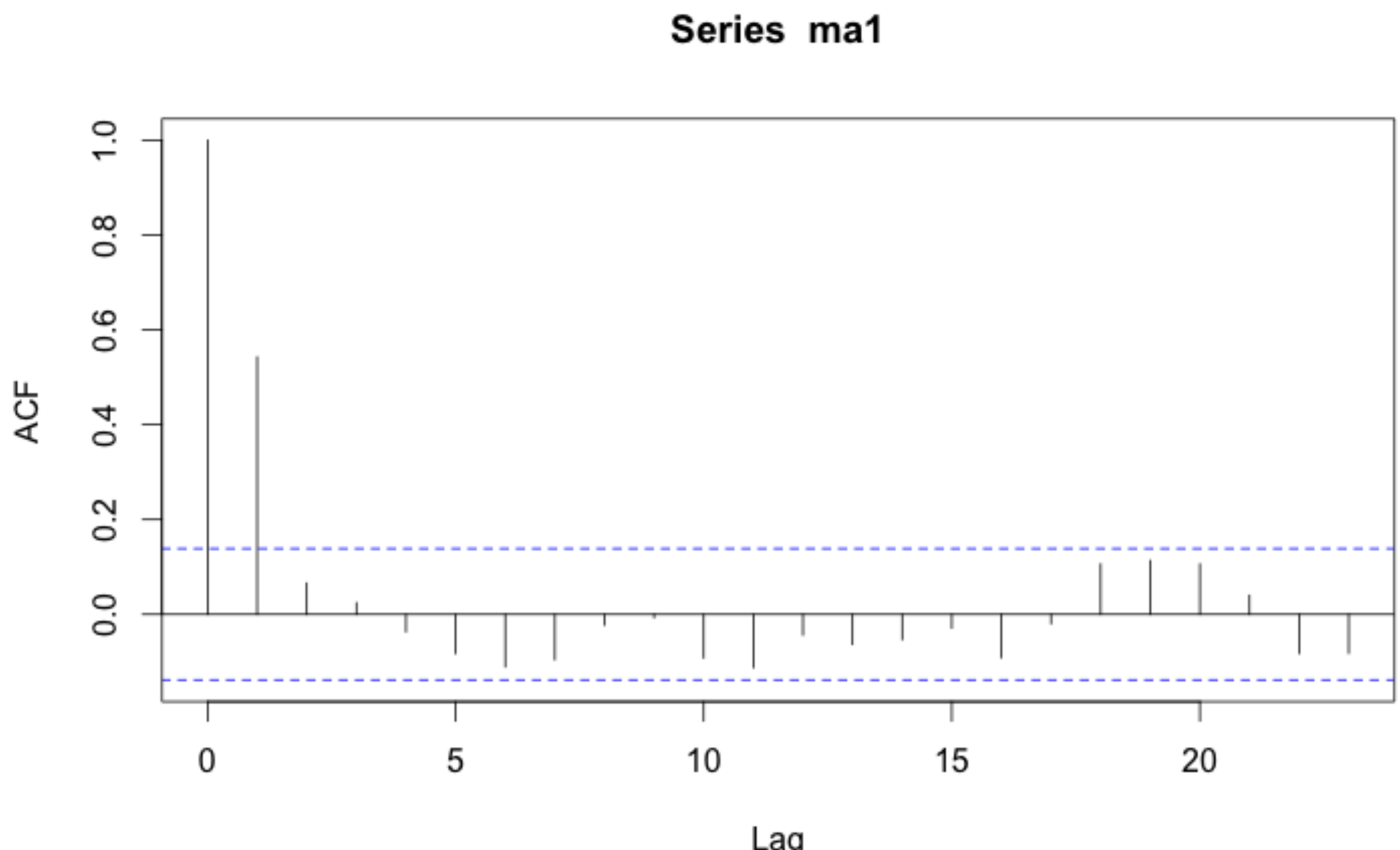
MA(1) $x_t = \beta_1 w_{t-1} + w_t$

What is the mean function?

What is the autocovariance function?

Is MA(1) stationary?

MA(1) $\beta_1 = 1$



ACF for simulated data

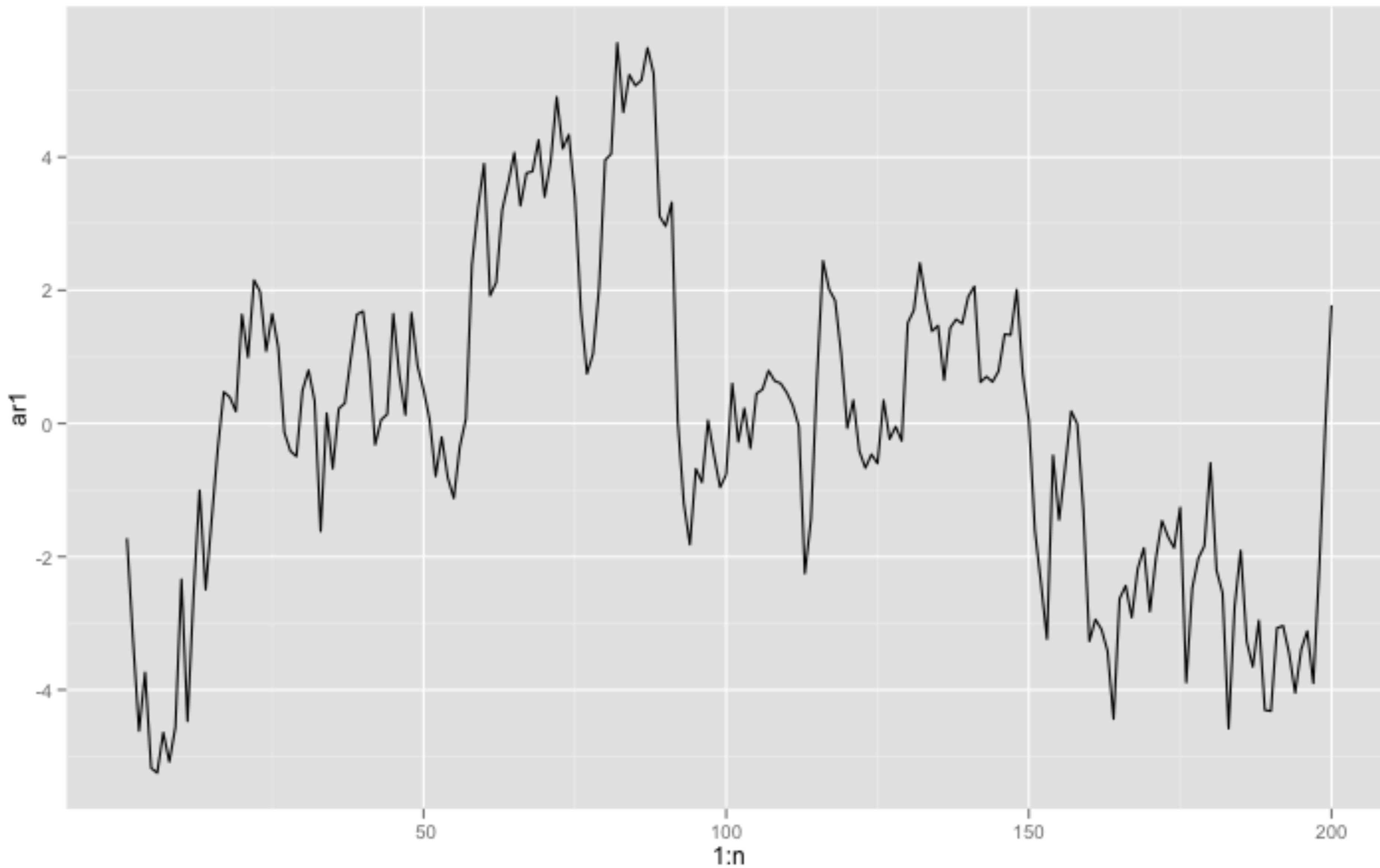
Autoregressive AR(1)

$$x_t = \alpha_1 x_{t-1} + w_t$$

where $\{w_t\}$ is a white noise process.

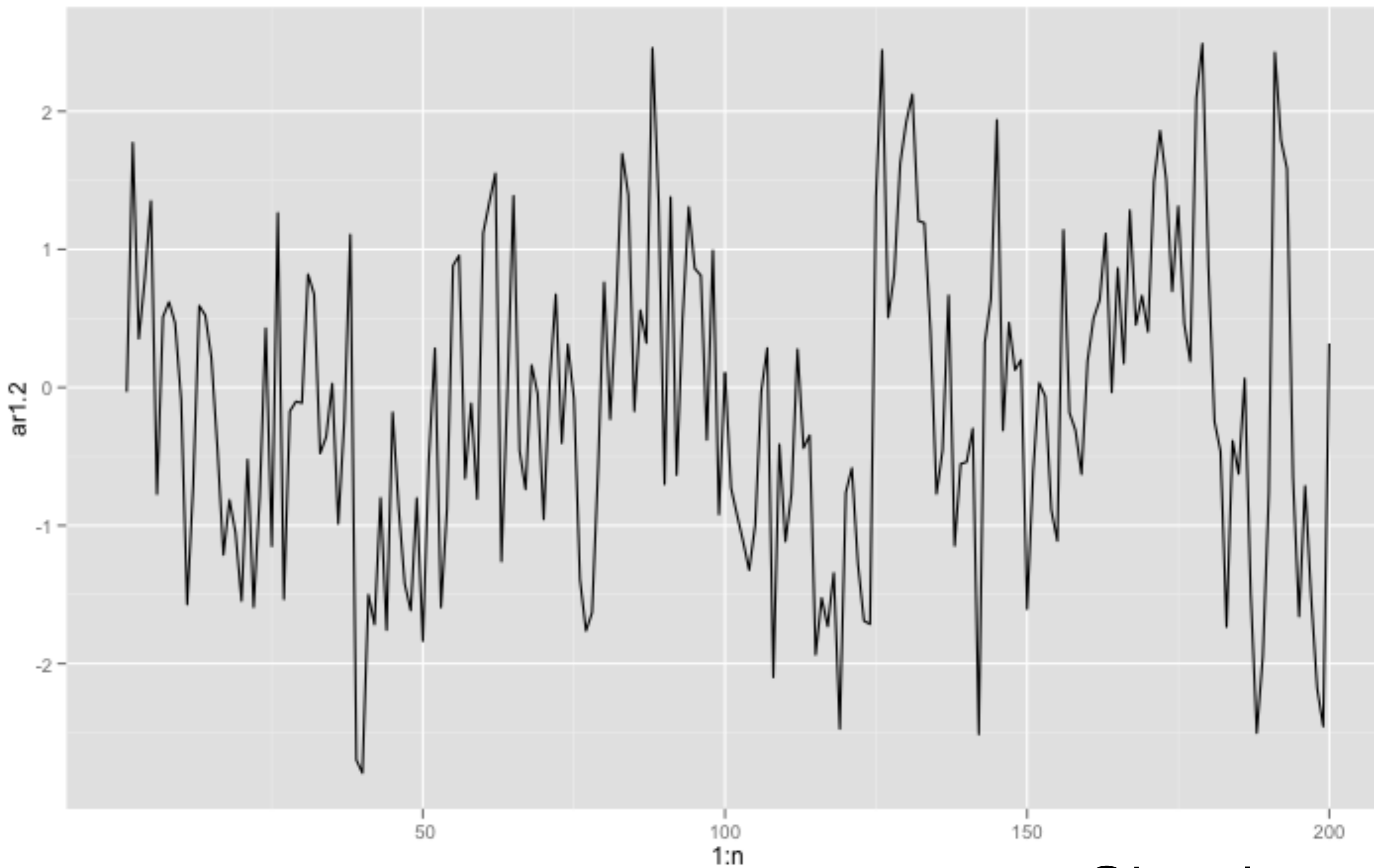
We'll see higher order AR processes later...

AR(1) $\alpha_1 = 0.9$



Simulated

AR(1) $\alpha_1 = 0.5$



Simulated

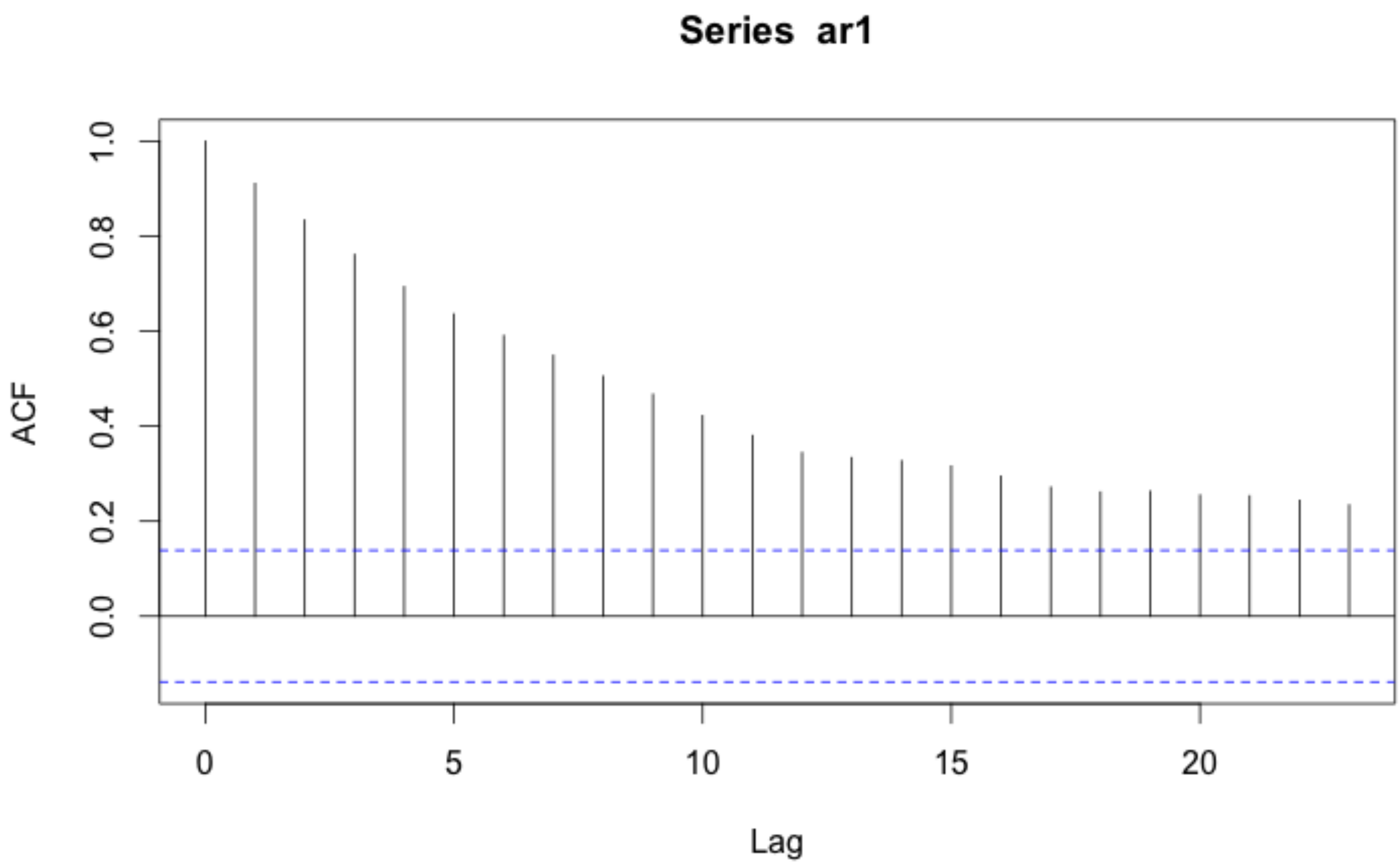
AR(1)

What is the mean function?

What is the autocovariance function?

Is AR(1) stationary?

AR(1) $\alpha_1 = 0.9$



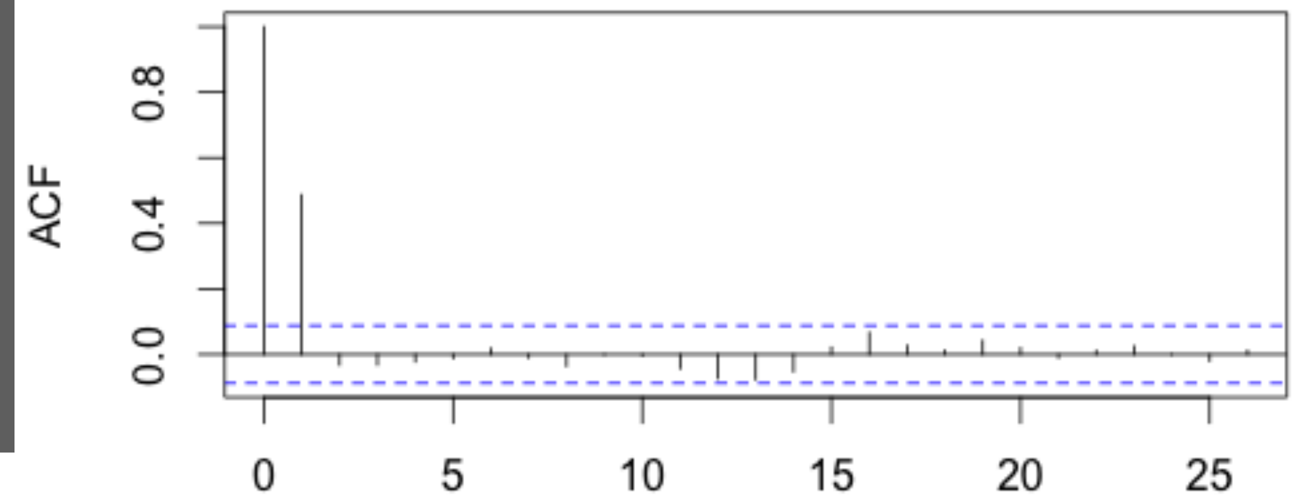
ACF for simulated data

Three stationary models

White noise	MA(1), any β_1	AR(1), $\|\alpha_1\| < 1$
$\rho(h) = 1$, when $h = 0$ $= 0$, otherwise	$\rho(h) = 1$, when $h = 0$ $= \beta_1/(1 + \beta_1^2)$, $h = 1$ $= 0$, $h \geq 2$	$\rho(h) = 1$, when $h = 0$ $= \alpha_1^h$, $h > 0$
Only lag 0 shows non-zero ACF.	Only lag 0 and 1 show non-zero ACF.	Decreasing ACF

Which models might these simulated data come from?

1



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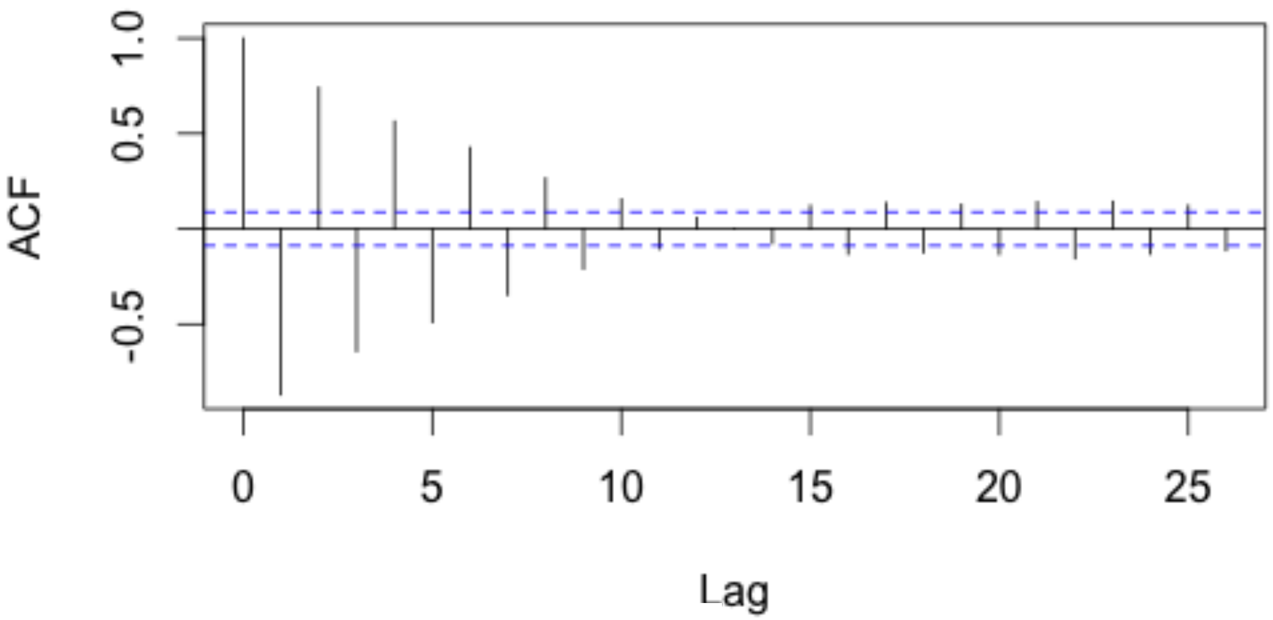

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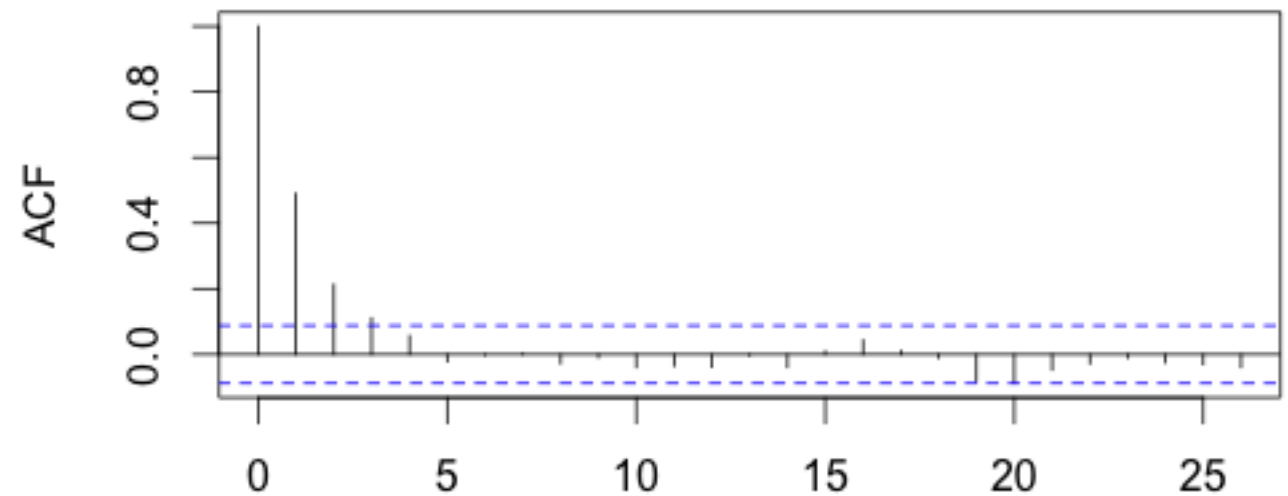

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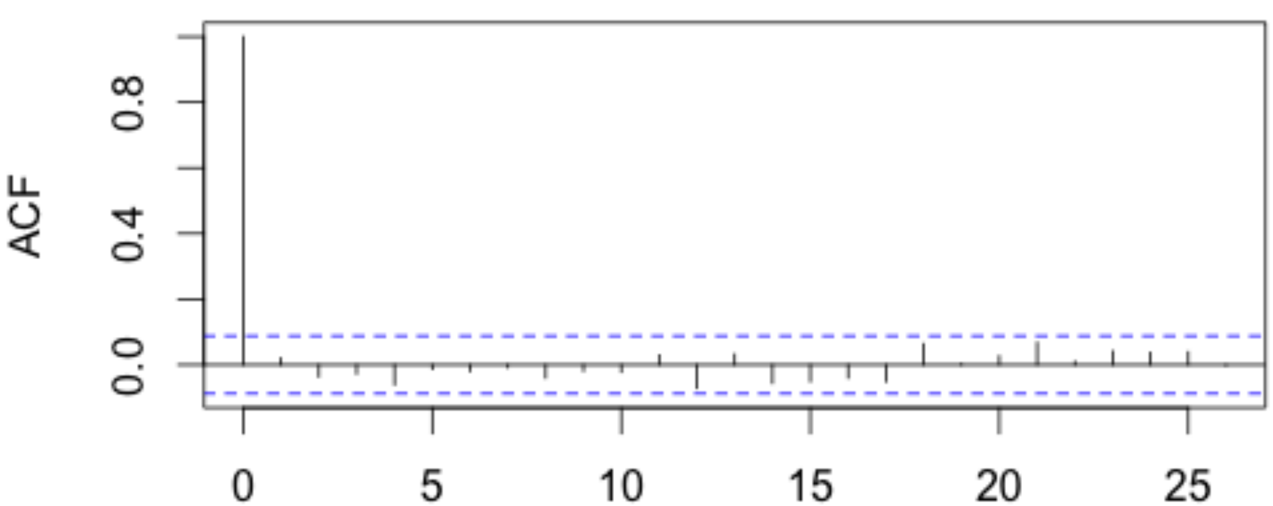

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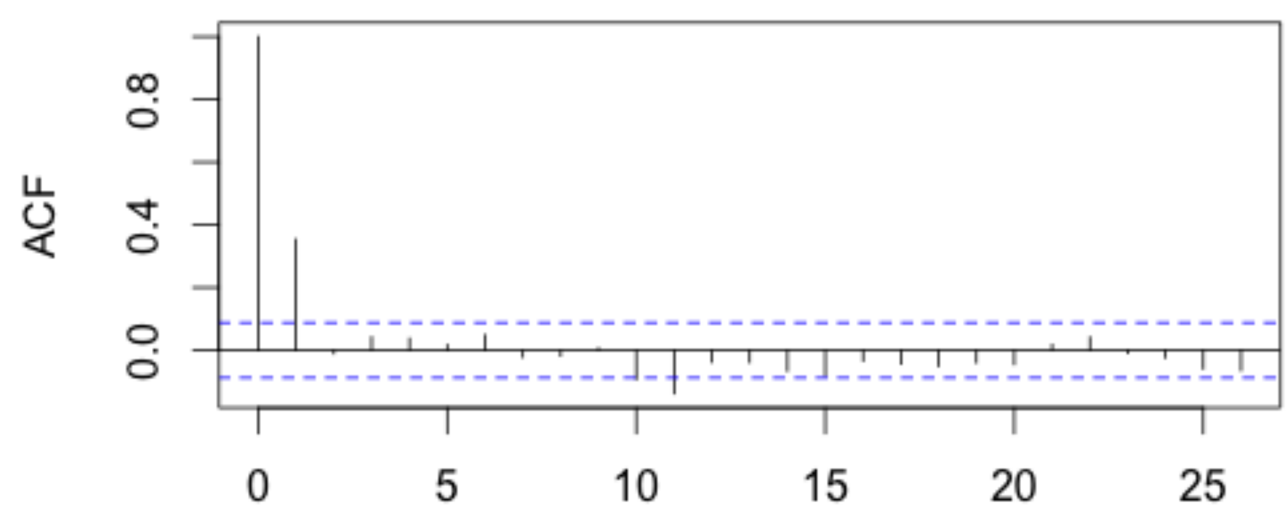

A General Linear Process

A linear process x_t is defined to be a linear combination of white noise variates, Z_t,

$$x_t = \sum_{i=0}^{\infty} \psi_i Z_{t-i}$$

with

$$\sum_{i=0}^{\infty} |\psi_i| < \infty$$

This is enough to ensure stationarity

Autocovariance

One can show that the autocovariance of a linear process is,

$$\gamma(h) = \sigma^2 \sum_{i=0}^{\infty} \psi_{i+h}\psi_i$$

Your turn

Write the MA(1) and AR(1) processes in the form of linear processes.

I.e. what are the ψ_j?

$$x_t = \sum_{i=0}^{\infty} \psi_i Z_{t-i}$$

Verify the autocovariance functions for MA(1) and AR(1)

$$\gamma(h) = \sigma^2 \sum_{i=0}^{\infty} \psi_{i+h} \psi_i$$

MA(1) we did
AR(1) you do

Backshift Operator

The **backshift** operator, B, is defined as

$$Bx_t = x_{t-1}$$

It can be extended to powers in the obvious way:

$$B^2x_t = (BB)x_t = B(Bx_t) = Bx_{t-1} = x_{t-2}$$

So, $B^kx_t = x_{t-k}$

MA(1): $x_t = \beta_1 Z_{t-1} + Z_t$

AR(1): $x_t = \alpha_1 x_{t-1} + Z_t$

Your turn

Write the MA(1) and AR(1) models using the backshift operator.

Difference Operator

The **difference** operator, ∇, is defined as,

$$\nabla^d x_t = (1 - B)^d x_t$$

$$(\text{e.g. } \nabla^1 x_t = (1 - B)\, x_t = x_t - x_{t-1})$$

$(1\text{-}B)^d$ can be expanded in the usual way,

$$\text{e.g. } (1 - B)^2 = (1 - B)(1 - B) = 1 - 2B + B^2$$

Some non-stationary series can be made stationary by differencing, see HW#2.

Roadmap

Extend AR(1) to AR(p) and MA(1) to MA(q)

Combine them to form ARMA(p, q) processes

Discover a few hiccups, and resolve them.

Then find the ACF (and PACF) functions for ARMA(p, q) processes.

Figure out how to fit a ARMA(p,q) process to real data.